1 Basics

\[ \ddot{a}_m = (1 + i) a_m \rightarrow \ddot{s}_m = (1 + i) s_m \]

\[ \delta^* = 2\delta \rightarrow \nu^* = \nu^2; \ i^* = 2i + i^2; \ d^* = 2d - d^2 \]

\[ t p_x = \exp \left( - \int_x^{x+t} \mu_s ds \right) \]

Under DML: \[ \mu_{x+t} = \frac{1}{\omega - x - t}; \ t p_x \mu_{x+t} = \frac{1}{\omega - x} \]

\[ e_x = \sum_{k=1}^{\infty} kp_x; \text{ under CF } = \frac{p_x}{q_x} \]

Under UDD (and therefore DML): \[ \hat{e}_x = e_x + .5 \]

Under DML: \[ \hat{e}_{x:m} = (n) n p_x + \left( \frac{n}{2} \right) n q_x \]
2 Insurances

\[ A_{x:m} = A^1_{x:m} + nE_x \]

\[ A_x = A^1_{x:m} + nE_x A_{x+n} \to nE_x = \frac{A_x - A^1_{x:m}}{A_{x+n}} ; A_{x+n} = \frac{A_x - A^1_{x:m}}{nE_x} \]

\[(IA)_x = A_x + \nu p_x (IA)_{x+1} \]

\[(DA)^1_{x:m} = n^\nu q_x + \nu p_x (DA)^1_{x+1,m-1} \]

\[ A_x = \nu q_x + \nu p_x A_{x+1} = \nu q_x + \nu^2 1q_x + \nu^2 2p_x A_{x+2} \]

\[ A_{x:m} = n_\gamma A_{x:m} + nE_x \]

\[ n_\gamma A_x = nE_x \bar{A}_{x+n} \]

2.1 Under CF

\[(\bar{I} \bar{A})_x = \bar{A}_x \bar{a}_x = \frac{\mu}{(\mu + \delta)^2} \]

\[ \bar{A}_{x:m} = \bar{A}_x (1 - nE_x) + nE_x \]

\[ A^1_{x:m} = A_x (1 - nE_x) = \frac{q}{q + t} (1 - nE_x) \]

\[ A^1_{x:m} = \bar{A}_x (1 - nE_x) \]

2.2 Under DML

\[ \bar{A}_x = \frac{\bar{a}_{\omega - x}}{\omega - x} \]

\[ A_x = \frac{a_{\omega - x}}{\omega - x} \]

\[ A^1_{x:m} = \frac{a_{\omega - x}}{\omega - x} \]
3 Annuities

\[ \ddot{a}_{x:m} = \frac{1 - A_{x:m}}{d} \]

\[ A_{x:m} = 1 - d \ddot{a}_{x:m} \]

\[ \ddot{a}_{x:m} = \sum_{k=0}^{n} \nu^k k p_x \]

\[ \ddot{a}_x = 1 + \nu p_x \ddot{a}_{x+1} \]

\[ \ddot{a}_{x:m} + n E_x = a_{x:m} + 1 \]

\[ \ddot{a}_x = \ddot{a}_{x:m} + n \ddot{a}_x \]

\[ n | \ddot{a}_x = n E_x \ddot{a}_{x+n} = \ddot{a}_x - \ddot{a}_{x:m} \]

\[ \ddot{a}_{x:m} = \ddot{a}_m + n | \ddot{a}_x = \ddot{a}_x + \ddot{a}_m - \ddot{a}_{x:m} \]

\[ \text{Var}(Y = \ddot{a}_{T(x)\wedge m}) = \frac{2 \bar{A}_{x:m} - (\bar{A}_{x:m})^2}{\delta^2} \quad \text{(analogous to \(\ddot{a}_{x:m}\))} \]

3.1 Under CF

\[ \ddot{a}_x = \ddot{a}_{x:m} + n | \ddot{a}_x = \ddot{a}_x (1 - n E_x) + n E_x \ddot{a}_{x+n} \]

\[ \ddot{a}_{x:m} = \frac{1 + i}{q + i} (1 - n E_x) \]

4 Miscellaneous

4.1 Payable \(m\)th-ly

\[ \left( 1 - \frac{d^{(m)}}{m} \right)^m = 1 - d \rightarrow d = 1 - \left( 1 - \frac{d^{(m)}}{m} \right)^m \]

\[ \ddot{a}_x < \ddot{a}_x^{(m)} < \ddot{a}_x \]

3
\[ A_x^{(m)} = 1 - d^{(m)} \ddot{a}_x^{(m)} \]
\[ \ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m - 1}{2m} \]
\[ \ddot{a}_x^{(m)} = \frac{1}{d^{(m)}} \]

4.1.1 Under UDD

\[ \ddot{a}_x^{(m)} = \alpha(m) \ddot{a}_x^{m} - \beta(m) (1 - n E_x) \]  ($1$ when pmts start - $1$ when pmts end)
\[ A_x^{(m)} = \frac{i}{\dot{i}} A_x \]

4.2 Statistics and Percentiles

Minimum sufficient amount = \( n F E[Z] + \phi(Pr) \sqrt{n F^2 Var(Z)} \)

For mixtures of RVs:  \( E[Y] = \sum_i w_i E[Y_i] \) ; \( E[Y^2] = \sum_i w_i E[Y_i^2] \) ; ... 

100\(p\)'th percentile of the PV RV = \( \nu^h \) for insurances, \( \bar{a}_T \) for annuities.

For insurances: \( d p_x - h p_x = 1 - p \) (where \( d \) = deferral period)

For annuities: \( d p_x - h p_x = p \)

Under CF, \( Pr(\bar{a}_T > \ddot{a}_x) = Pr(\nu^T < \ddot{A}_x) = Pr(\nu^T < \mu \ddot{a}_T) = \left( \frac{\mu}{\mu + \delta} \right)^{\mu/\delta} \)

5 Premiums

\( E[L] = A_x - P \ddot{a}_x \)

Under EP: \( P_x = \frac{1}{\ddot{a}_x} - d = \frac{d A_x}{1 - A_x} \)
For \( n \)-pay whole life: 
\[ n\bar{P}_x = \frac{A_x}{\bar{a}_{x:m}} \]

Under CF: 
\[ P_x = \bar{P}_x^1 = \nu \mu ; \quad \bar{P}_x = \bar{P}_x^1 = \mu \]

\[ Var(L) = \left(1 + \frac{\bar{P}}{\delta}\right)\left(\tilde{A}_x - (\tilde{A}_x)^2\right) \quad \text{under EP} = \frac{2\tilde{A}_x - (\tilde{A}_x)^2}{(1 - A_x)^2} \]

For FCWL under EP, CF: 
\[ Var(L) = 2\tilde{A}_x \]

6 Reserves

6.1 Standard Formulas

\[ \bar{P}_x = 1 - d \rightarrow \bar{a}_{x:m} = \frac{1}{d + P_x}\]

\[ P_x = \frac{d A_x}{1 - A_x} \rightarrow A_{x:m} = \frac{P_x}{d + P_x} \]

Prospective formula: 
\[ tV_{x:m} = PVFB - PVFP = A_{x+t,\bar{n} - \bar{n}} - P_x\bar{a}_{x+t,\bar{n} - \bar{n}} \]

The following three formulas hold under EP:

- Annuity-ratio: 
  \[ tV_{x:m} = 1 - \frac{\bar{a}_{x+t,\bar{n} - \bar{n}}}{\bar{a}_{x:m}} \]

- Insurance-ratio: 
  \[ tV_{x:m} = \frac{A_{x+t,\bar{n} - \bar{n}} - A_x}{1 - A_{x:m}} \]

- Premium-ratio: 
  \[ tV_{x:m} = \frac{P_{x+t,\bar{n} - \bar{n}} - P_x}{P_{x+t,\bar{n} - \bar{n}} + d} \]

Retrospective formula: 
\[ tV_{x:m} = AVPP - AVPB = P_x \bar{s}_{x:m} - tk_x \]

where \( tk_x = \text{accumulated cost of insurance} = \frac{A_{x,\bar{n}}}{tE_x} \)
6.2 Other Formulas

Under UDD: \( nV(\bar{A}_x) = \bar{A}_{x+n} - P(\bar{A}_x) \bar{a}_{x+n} = \frac{i}{\delta} nV_x \)

For WL or term under CF: \( kV = 0 \)

\[
(kV + \pi_k) (1 + i_{k+1}) = q_{x+k} b_{k+1} + p_{x+k} k_{k+1} V = k_{k+1} V + q_{x+k} (b_{k+1} - k_{k+1} V)
\]

Initial benefit reserve for year \( k = k-1 V + \pi_{k-1} \)

Net amount at risk for year \( k = b_k - kV \)

\[
nV = P \hat{s}_m - \sum_{k=0}^{n-1} \nu q_{x+k} (b_{k+1} - k_{k+1} V) (1 + i)^{n-k}
\]

Hattendorf: \( \text{Var}(kL) \)

\[
\nu^2 (b_{k+1} - k_{k+1} V)^2 p_{x+k} q_{x+k} + \nu^4 (b_{k+2} - k_{k+2} V)^2 p_{x+k+1} q_{x+k+1} p_{x+k} + \nu^6 (b_{k+3} - k_{k+3} V)^2 p_{x+k+2} q_{x+k+2} 2p_{x+k} + ...
\]

6.3 \( \frac{P}{P} \) Problems

- Accumulated difference of premiums = difference in reserves

\[
P_x \hat{s}_{x:m} = nV_x + n k_x = nV_x + n k_x \quad (1)
\]

\[
P_{x:m} \hat{s}_{x:m} = nV_{x:m} + n k_x = 1 + n k_x \quad (2)
\]

\[
P_{1:x:m} \hat{s}_{x:m} = nV_{1:x:m} + n k_x = 0 + n k_x \quad (3)
\]

\[
nP_x \hat{s}_{x:m} = nV_x + n k_x = A_{x+n} + n k_x \quad (4)
\]

(4) - (3): \( (nP_x - P_{1:x:m}) \hat{s}_{x:m} = nV_x - nV_{1:x:m} = A_{x+n} \)

(2) - (4): \( (P_{x:m} - nP_x) \hat{s}_{x:m} = nV_{x:m} - nV_x = 1 - A_{x+n} \)

(2) - (1): \( (P_{x:m} - P_x) \hat{s}_{x:m} = nV_{x:m} - nV_x = 1 - nV_x \)

(1) - (3): \( (P_x - P_{1:x}) \hat{s}_{x:m} = nV_{x:m} - nV_x = nV_x \)

- \( P_{x:m} \hat{s}_{x:m} = \frac{n E_x}{\hat{a}_{x:m}} \hat{s}_{x:m} = 1 \rightarrow \text{Replace } \hat{s}_{x:m} \text{ with } \frac{1}{P_{x:m}} \text{ above.} \)

\( (nV_x = k \text{th reserve for } n \text{-pay WL} = A_{x+k} - nP_x \bar{a}_{x+k; n-k}) \)
7 Multiple Lives

\[ nq_{xy} = \int_0^n np_x np_y \mu_{x+t} \mu_{y+t} dt \quad \rightarrow \quad nq_{xx} = \int_0^n (np_x)^2 2 \mu_{x+t} dt \]

\[ n|mq_{xy} = n + nq_{xy} - nq_{xy} = n + mq_x n + mq_y - nq_x nq_y \]

\[ \hat{e}_{xy} = \int_0^\infty t p_x t p_y dt \]

\[ \hat{e}_x + \hat{e}_y = \hat{e}_{xy} + \hat{e}_{xy} \]

\[ A_{xy} \text{ with common shock} = A_x[\mu=\mu_x+\mu_c] + A_y[\mu=\mu_y+\mu_c] - A_{xy}[\mu=\mu_x+\mu_y+\mu_c] \]

\[ \text{CoV}(T(xy), T(\overline{xy})) = E[T(xy) T(\overline{xy})] - \hat{e}_{xy} \hat{e}_{xy} = (\hat{e}_x - \hat{e}_{xy})(\hat{e}_y - \hat{e}_{xy}) \]

7.1 Contingent Functions

\[ nq_{xy}^1 = \Pr(T(x) < T(y) \cap T(x) < n) = \int_0^n t p_{xy} \mu_{x+t} dt \]

\[ nq_{xy}^2 = \Pr(T(y) < T(x) < n) = \int_0^n t p_x t q_y dt \]

7.1.1 Identities

\[ nq_{xy}^1 + nq_{xy}^1 = nq_{xy} \]

\[ nq_{xy}^2 + nq_{xy}^2 = nq_{xy} = nq_x nq_y \]

\[ nq_{xy}^1 + nq_{xy}^2 = nq_x \]

7.1.2 Under DML

\[ nq_{xy}^1 = nq_x n/2 p_y \quad (x \text{ dies})(y \text{ still alive when } x \text{ expected to die}) \]

\[ nq_{xy}^2 = nq_x nq_y \frac{1}{2} \quad \text{(both die)(right order)} \]
7.1.3 Under CF

\[ q_{xy}^1 = \mu_x \hat{e}_{xy} = \frac{\mu_x}{\mu_x + \mu_y} \]

\[ nq_{xy}^1 = \mu_x \hat{e}_{xy} = \mu_x (\hat{e}_{xy} (1 - n p_{xy})) = \frac{\mu_x}{\mu_x + \mu_y} n q_{xy} = q_{xy}^1 n q_{xy} \]

7.2 Contingent Insurances

\[ \bar{A}_{xy}^1 = \int_0^\infty \nu^t p_{xy} \mu_{x+t} dt \]

\[ \bar{A}_{xy}^2 = \int_0^\infty \nu^t p_{xy} \mu_{x+t} q_y dt = \int_0^\infty \nu^t p_{xy} \mu_{y+t} \bar{A}_{x+t} dt \]

7.2.1 Identities

\[ \bar{A}_x = \bar{A}_{xy}^1 + \bar{A}_{xy}^2 \]

\[ \bar{A}_{xy} = \bar{A}_{xy}^1 + \bar{A}_{xy}^1 \]

\[ \bar{A}_{xy} = \bar{A}_{xy}^2 + \bar{A}_{xy}^2 \]

7.2.2 Under DML

\[ \bar{A}_{xy}^1 = q_x \bar{a}_y \]

7.2.3 Under CF

\[ \bar{A}_{xy}^1 = \frac{\mu_x}{\mu_x + \mu_y + \delta} \]

\[ \bar{A}_{xy}^2 = \int_0^\infty \nu^t p_{xy} \mu_{x+t} \bar{A}_{y+t} dt = \bar{A}_{xy}^1 \bar{A}_y \] (\bar{A}_y when x dies, if x dies first)
7.3 Reversionary Annuities

\[ \bar{a}_{x|y} (\bar{a}_{\text{failing}|\text{surviving}}) = \bar{a}_y - \bar{a}_{xy} \] (\( y \) gets paid after \( x \) dies)

\[ \bar{a}_{x|y|m} = \bar{a}_{y|m} - \bar{a}_{xy|m} \]

\[ \bar{a}_{m|x} = n\bar{a}_x = nE_x \bar{a}_{x+n} \] (\( m \) = failing status)

8 Multiple Decrements

\[ \prod_j p_x^{(j)} = p_x^{(\tau)} = 1 - q_x^{(\tau)} = 1 - \sum_j q_x^{(j)} \] (even given no assumptions)

\[ q_x^{(j)} = 1 - p_x^{(j)}; \quad p_x^{(j)} = \exp \left(- \int_0^1 \mu_x^{(j)} \, dt \right) \]

\[ \ell_x^{(j)} = \sum_{k=x}^{\infty} d_k^{(j)} \rightarrow \ell_x^{(\tau)} = \sum_j \ell_x^{(j)} \]

\[ nq_x^{(j)} = \sum_{k=0}^{n-1} \frac{d_k^{(j)}}{\ell_x^{(j)}} = \int_0^n q_x^{(j)} \mu_{x+t} \, dt \] (affected by changes in any decrement)

If the forces of each decrement are proportional, then the \( q_x^{(j)} \)'s have the same proportions relative to \( q_x^{(\tau)} \).

8.1 Density Functions

\[ f_{T,J}(t,j) = q_x^{(\tau)} \mu_{x+t} \]

\[ f_J(j) = Pr(J = j) = \frac{\ell_x^{(j)}}{\ell_x^{(\tau)}} = \int_0^\infty q_x^{(j)} \mu_{x+t} \, dt = \infty \]

\[ f_T(t) = q_x^{(\tau)} \mu_{x+t} \]

\[ f_{J|T}(j|t) = Pr(J = j \mid T = t) = \frac{\mu_{x+t}^{(j)}}{\mu_{x+t}^{(\tau)}} \]
8.2 Under UDDMDT

\[ p'_x = \left( p_x^{(r)} \right) \phi^{(r)} / \psi^{(r)} \]

8.3 Under UDDASDT

\[ \tau q'_x = (t) q_x^{(j)}, \quad 0 \leq t \leq 1, \text{ for all } j \]

For \( m = 2 \):

\[ q_x^{(1)} = q_x^{(1)} (1 - \frac{1}{2} q_x^{(2)}) \]

For \( m = 3 \):

\[ q_x^{(1)} = q_x^{(1)} (1 - \frac{1}{2} q_x^{(2)} - \frac{1}{2} q_x^{(3)} + \frac{1}{3} q_x^{(2)} q_x^{(3)}) \]

9 Expenses

\% of premium, \% of benefit amount, and fixed expenses are incurred at the beginning of each policy year (\% of premium expenses are only incurred while premium is still being paid.)

Settlement charges are incurred when the death benefit is paid.

Extended EP: \( E[0L_e] = 0 \rightarrow PVFB + PVFE = PVF(ELP) \)

where ELP = expense loaded premium = \( P \) + expense loading.

Expense-augmented reserve: \( kV^E = E[kL_e] = PVFB + PVFE - PVF(ELP) \)

Expense reserve = PVFE - PVF (L = level expense loading to \( P_x \))

9.1 Asset Shares

Two decrements: \( d \) = death; \( w \) = withdrawal

\( (k AS + G(1 - c_k) - e_k)(1 + i) = b_{k+1} q_{x+k}^{(d)} + k+1 CV q_{x+k}^{(w)} + k+1 AS p_{x+k}^{(r)} \)

\( nAS = \sum_{k=0}^{n-1} \left[ G(1 - c_k) - e_k - b_{k+1} \nu q_{x+k}^{(d)} - k+1 CV \nu q_{x+k}^{(w)} \right] \frac{(1 + i)^{n-k}}{n-k p_{x+k}^{(r)}} \)
10 Poisson Processes

\[ Pr(\text{Poisson}(\lambda) = k) = e^{-\lambda} \frac{\lambda^k}{k!} \]

\[ m(t) = \int_0^t \lambda(x) \, dx ; \text{effective } \lambda \text{ between } a \text{ and } b = m(b) - m(a) \]

Thinned Poisson “sub-processes” are independent.

10.1 Compound Poisson Processes

Let \( S(t) = X_1 + X_2 + \ldots + X_{N(t)} \) be a process such that frequency follows a Poisson process \( N(t) \), and severity is represented by \( X_i \) (IID per event). Then:

- \( E[S] = \lambda N E[X_i] \)
- \( Var(S) = \lambda N E[X_i^2] = \lambda N (Var(X_i) + (E[X_i])^2) \)

10.2 Mixed Poisson Processes and Negative Binomial

Suppose \( N(t) \) is a Poisson process with a rate that is Gamma-distributed with parameters \( (\alpha, \theta) \). Then:

\( N(t) \sim NB(r = \alpha, \beta = t \theta) \)

Recall:

- \( X \sim \Gamma(\alpha, \theta) \rightarrow E[X] = \alpha \theta ; \, Var(X) = \alpha \theta^2 \)
- \( X \sim NB(r, \beta) \rightarrow E[X] = r \beta ; \, Var(X) = r \beta (\beta + 1) \)

Suppose \( X \sim NB(r, \beta) \). Let \( t = r, \, b = \beta, \, x = k \). Then:

\[ Pr(X = k = x) = \left( \frac{t + x - 1}{x} \right) \left( \frac{1}{1 + b} \right)^t \left( \frac{b}{1 + b} \right)^x \]
11 Markov Chains

$Q = \text{transition matrix for 1 period (homogeneous)}$

$kQ = \text{transition matrix for k periods } = Q^k$ (homogeneous)

$Q_n = \text{transition matrix from time } n \text{ to time } n + 1$ (non-homogeneous)

$Q^{(i,j)} = Pr(\text{in state } j \text{ next period } | \text{ in state } i \text{ now})$ \quad (Q^{(row,column)})

$kQ^{(i,j)} = Pr(\text{in state } j \text{ k periods from now } | \text{ in state } i \text{ now})$

$P^{(i)} = Q^{(i,i)} = Pr(\text{stay in state } i \text{ for 1 period})$

$kP^{(i)} = (P^{(i)})^k = Pr(\text{stay in state } i \text{ for k periods})$ \quad (\neq kQ^{(i,i)})

For Markov-modeled insurance, keep in mind that you can only die once (so the probability that the insured was already dead must be subtracted out from each probability vector).