Section 13, Zero-Truncated Distributions

Frequency distributions can be constructed that have support on the positive integers or alternately have \( f(0) = 0 \). For example, let \( f(x) = \frac{(e^{-3/3^x}/x!)}{(1 - e^{-3})} \) for \( x = 1, 2, 3, \ldots \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>15.719%</td>
<td>23.578%</td>
<td>23.578%</td>
<td>17.684%</td>
<td>10.610%</td>
<td>5.305%</td>
<td>2.274%</td>
</tr>
<tr>
<td>( F(x) )</td>
<td>15.719%</td>
<td>39.297%</td>
<td>62.875%</td>
<td>80.558%</td>
<td>91.169%</td>
<td>96.4736%</td>
<td>98.74718%</td>
</tr>
</tbody>
</table>

Exercise: Verify that the sum of \( f(x) = \frac{(e^{-3/3^x}/x!)}{(1 - e^{-3})} \) for \( x = 1 \) to \( \infty \) is unity.

[Solution: The sum of the Poisson Distribution from 0 to \( \infty \) is 1. \( \sum_{i=0}^{\infty} e^{-3/3^x}/x! = 1 \).]

Therefore, \( \sum_{i=1}^{\infty} e^{-3/3^x}/x! = 1 - e^{-3} \Rightarrow \sum_{i=1}^{\infty} f(x) = 1 \).

This is an example of a Poisson Distribution Truncated from Below at Zero, with \( \lambda = 3 \).

In general, if \( f \) is a distribution on 0, 1, 2, 3,..., then \( h(x) = f(x) / \{1 - f(0)\} \) is a distribution on 1, 2, 3, ...

This is a special case of truncation from below. The general concept of truncation of a distribution is covered in a “Mahler’s Guide to Loss Distributions.”

We have the following three examples, shown in Appendix B.3.1 of Loss Models:

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Density of the Zero-Truncated Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binomial</td>
<td>( \frac{m! ; q^x ; (1 - q)^{m-x}}{x! ; (m-x)! ; 1 - (1-q)^m} ) ( x = 1, 2, 3,..., m )</td>
</tr>
<tr>
<td>Poisson</td>
<td>( \frac{e^{-\lambda} ; \lambda^x ; / ; x!}{1 - e^{-\lambda}} ) ( x = 1, 2, 3,... )</td>
</tr>
<tr>
<td>Negative Binomial</td>
<td>( \frac{r(r+1)...(r+x-1) ; \beta^x}{x! ; (1+\beta)^x + r} ; 1 - 1/(1+\beta)^r} ) ( x = 1, 2, 3,... )</td>
</tr>
</tbody>
</table>

See Section 6.7 in Loss Models.
Moments:

Exercise: For a Zero-Truncated Poisson with $\lambda = 3$, what is the mean?

[Solution: Let $f(x)$ be the untruncated Poisson, and $h(x)$ be the truncated distribution. Then $h(x) = f(x) / \{1 - f(0)\}$.

The mean of $h = \sum_{x=1}^{\infty} x h(x) = \sum_{x=1}^{\infty} \frac{x f(x)}{1 - f(0)} = \frac{\sum_{x=0}^{\infty} x f(x)}{1 - f(0)} = \frac{\text{mean of } f}{1 - f(0)} = \frac{\lambda}{1 - e^{-\lambda}} = \frac{3}{1 - e^{-3}} = 3.157.]

In general, the moments of a zero-truncated distribution, $h$, are given in terms of those of the corresponding untruncated distribution, $f$, by: $E_h[X^n] = \frac{E_f[X^n]}{1 - f(0)}$.

For example for the Zero-Truncated Poisson the mean is: $\lambda / (1 - e^{-\lambda})$,
while the second moment is: $(\lambda + \lambda^2) / (1 - e^{-\lambda})$.

Exercise: For a Zero-Truncated Poisson with $\lambda = 3$ what is the second moment?

[Solution: Let $f(x)$ be the untruncated Poisson, and $h(x)$ be the truncated distribution. Then $h(x) = f(x) / \{1 - f(0)\}$.

The second moment of $f$ is its variance plus the square of its mean = $\lambda + \lambda^2$.
The second moment of $h$ = (the second moment of $f$) / $\{1 - f(0)\}$ =
$(\lambda + \lambda^2) / (1 - e^{-\lambda}) = (3 + 3^2)/(1 - e^{-3}) = 12.629.$]

Thus a Zero-Truncated Poisson with $\lambda = 3$ has a variance of $12.629 - 3.157^2 = 2.66$.
This matches the result of using the formula in Appendix B of Loss Models:
$\lambda \{1 - (\lambda+1)e^{-\lambda}\} (1 - e^{-\lambda})^2 = (3)\{1 - 4e^{-3}\} (1 - e^{-3})^2 = (3)(.8009)/(.9502)^2 = 2.66$.

It turns out that for the Zero-Truncated Negative Binomial, the parameter $r$ can take on values between -1 and 0, as well as the usual positive values, $r > 0$. This is sometimes referred to as the Extended Zero-Truncated Negative Binomial, however provided $r \neq 0$ all the same formulas apply. As $r$ approaches zero, the Zero-Truncated Negative Binomial approaches the Logarithmic Distribution.
The Logarithmic Distribution with parameter $\beta$ has support equal to the positive integers:

$$f(x) = \frac{\beta}{x \ln(1+\beta)} x^{\beta - 1}, \text{ for } x = 1, 2, 3,...$$

with mean: $\frac{\beta}{\ln(1+\beta)}$, and variance: $\frac{1 + \beta - \frac{\beta}{\ln(1+\beta)}}{\ln(1+\beta)}$.

Exercise: Assume the number of vehicles involved in each automobile accident is given by

$$f(x) = \frac{0.2^x}{\{x \ln(1.25)\}}, \text{ for } x = 1, 2, 3,...$$

Then what is the mean number of vehicles involved per automobile accident?

[Solution: This is a Logarithmic Distribution with $\beta = .25$, and mean $\frac{\beta}{\ln(1+\beta)} = .25/ \ln(1.25) = 1.12.$]

The density function of this Logarithmic Distribution with $\beta = .25$ is as follows:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>89.6284%</td>
<td>8.9628%</td>
<td>1.1950%</td>
<td>0.1793%</td>
<td>0.0287%</td>
<td>0.0048%</td>
<td>0.0008%</td>
</tr>
<tr>
<td>$F(x)$</td>
<td>89.628%</td>
<td>98.591%</td>
<td>99.786%</td>
<td>99.966%</td>
<td>99.994%</td>
<td>99.9990%</td>
<td>99.9998%</td>
</tr>
</tbody>
</table>

Exercise: Show that the densities of a Logarithmic Distribution sum to one.

[Solution: $\ln(1+\beta) = -\ln[1/(1+\beta)] = -\ln[1 - \beta/(1+\beta)] = -((-\beta/(1+\beta)) - (-\beta/(1+\beta))^2/2 + (-\beta/(1+\beta))^3/3 - (-\beta/(1+\beta))^4/4 + \ldots)$ = $((\beta/(1+\beta)) + (\beta/(1+\beta))^2/2 + (\beta/(1+\beta))^3/3 + (\beta/(1+\beta))^4/4 + \ldots = \Sigma[\beta/(1+\beta)]^x / x)].$

$f(x) = \{\beta/(1+\beta)\}^x / \{x \ln(1+\beta)\}$, $\Sigma f(x) = \Sigma[\beta/(1+\beta)]^x / x / \ln(1+\beta) = 1.$

Comment: $\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$, for $|x| \leq 1.$]

For the Logarithmic Distribution is: $P(z) = 1 - \frac{\ln[1 - \beta(z - 1)]}{\ln(1+\beta)}$. 


Exercise: Show that the limit as $r \to 0$ of Zero-Truncated Negative Binomial Distributions with the other parameter $\beta$ fixed, is a Logarithmic Distribution with parameter $\beta$.

Solution: For the Zero-Truncated Negative Binomial Distribution:

$$f(x) = \frac{(r+x-1)! \beta^x}{\{1+(1+\beta)^{x+r}\}} \div \{1-1/(1+\beta)^r\}$$

$$= \left\{r(r+1)...(r+x-1) / x!\right\} \{\beta/(1+\beta)\}^x \div \{(1+\beta)^r - 1\}.$$

$$\lim_{r \to 0} f(x) = \left\{\beta/(1+\beta)\right\} \div x! \lim_{r \to 0} \left\{(r+1)...(r+x-1) / x!\right\} \{r/(1+\beta)^r - 1\} =$$

$$\lim_{r \to 0} 1/\ln(1+\beta) \div \ln(1+\beta).$$

Where I have used L'Hospital's Rule.

This is the density of a Logarithmic Distribution.

Alternately, the p.g.f. of a Zero-Truncated Negative Binomial Distribution is:

$$P(z) = \left\{(1 - \beta(z-1))^{r} - (1+\beta)^{r}\right\}/\left\{1 - (1+\beta)^{r}\right\}.$$

$$\lim_{r \to 0} P(z) = \lim_{r \to 0} \left\{(1 - \beta(z-1))^{r} - (1+\beta)^{r}\right\}/\left\{1 - (1+\beta)^{r}\right\} =$$

$$\lim_{r \to 0} [- \ln(1-\beta(z-1))\{1 - \beta(z-1)\}^{r} + \ln(1+\beta)(1 + \beta)^{r}] / \{\ln(1+\beta) (1 + \beta)^{r}\} =$$

$$\lim_{r \to 0} \{\ln(1+\beta) - \ln[1 - \beta(z-1)]\} / \ln(1+\beta) = 1 - \ln[1 - \beta(z-1)] / \ln(1+\beta).$$

Where I have used L'Hospital's Rule.

This is the p.g.f of a Logarithmic Distribution.

(a,b,1) Class.\textsuperscript{91}

The (a,b,1) class of frequency distributions in Loss Models is a generalization of the (a,b,0) class. As with the (a,b,0) class, the recursion formula: $f(x)/f(x-1) = a + b/x$ applies. However, this relationship need only apply now for $x \geq 2$, rather than $x \geq 1$.

Members of the (a,b,1) family include: all the members of the (a,b,0) family,\textsuperscript{92} the zero-truncated versions of those distributions: Zero-Truncated Binomial, Zero-Truncated Poisson, and Extended Truncated Negative Binomial,\textsuperscript{93} and the Logarithmic Distribution.

In addition the (a,b,1) class includes the zero-modified distributions corresponding to these, to be discussed in the next section.

\textsuperscript{91} See Table 6.4 and Appendix B.3 in Loss Models.

\textsuperscript{92} Binomial, Poisson, and the Negative Binomial.

\textsuperscript{93} The Zero-Truncated Negative Binomial where in addition to $r > 0$, $-1 < r < 0$ is also allowed.
Probability Generating Functions:

The Probability generating function, \( P(z) = E[z^N] \), for a zero-truncated distribution can be obtained from that for the untruncated distribution.

\[
P^T(z) = \frac{P(z) - f(0)}{1 - f(0)},
\]

where \( P(z) \) is the p.g.f. for the untruncated distribution and \( P^T(z) \) is the p.g.f. for the zero-truncated distribution, and \( f(0) \) is the probability at zero for the untruncated distribution.

[Solution: For the untruncated Poisson \( P(z) = e^{\lambda(z-1)} \), \( f(0) = e^{-\lambda} \). \( P^T(z) = \{P(z) - f(0)\} /{1 - f(0)} = \{e^{\lambda(z-1)} - e^{-\lambda}\} /{1 - e^{-\lambda}} = \{e^{\lambda z} - 1\} /{e^\lambda - 1}.\]

One can derive this relationship as follows:

\[
P^T(z) = \sum_{x=1}^{\infty} z^n h(n) = \sum_{x=1}^{\infty} \frac{z^n f(n) - f(0)}{1 - f(0)} = \frac{P(z) - f(0)}{1 - f(0)}.
\]

In any case, Appendix B of *Loss Models* displays the Probability Generating Functions for all of the Zero-Truncated Distributions.

Notation:

*Loss Models* uses the following notation:

\( p_k \) the density function of the untruncated frequency distribution at \( k \).

\( p^T_k \) the density function of the zero-truncated frequency distribution at \( k \).

\( p^M_k \) the density function of the zero-modified frequency distribution at \( k \).  

Exercise: Give a verbal description of the following terms: \( p_7 \), \( p^M_4 \), and \( p^T_6 \).

[Solution: \( p_7 \) is the density of the frequency at 7, \( f(7) \).

\( p^M_4 \) is the density of the zero-modified frequency at 4, \( f_M(4) \).

\( p^T_6 \) is the density of the zero-truncated frequency at 6, \( f_T(6) \).]

\(^a\) Zero-modified distributions will be discussed in the next section.
Problems:

13.1 (1 point) The number of persons injured in an accident is assumed to follow a Zero-Truncated Poisson Distribution with parameter $\lambda = 0.3$. Given an accident, what is the probability that exactly 3 persons were injured in it?
A. Less than 1.0%
B. At least 1.0% but less than 1.5%
C. At least 1.5% but less than 2.0%
D. At least 2.0% but less than 2.5%
E. At least 2.5%

Use the following information for the next four questions:

The number of vehicles involved in an automobile accident is given by a Zero-Truncated Binomial Distribution with parameters $q = 0.3$ and $m = 5$.

13.2 (1 point) What is the mean number of vehicles involved in an accident?
A. less than 1.8
B. at least 1.8 but less than 1.9
C. at least 1.9 but less than 2.0
D. at least 2.0 but less than 2.1
E. at least 2.1

13.3 (2 points) What is the variance of the number of vehicles involved in an accident?
A. less than .5
B. at least .5 but less than .6
C. at least .6 but less than .7
D. at least .7 but less than .8
E. at least .8

13.4 (1 point) What is the chance of observing exactly 3 vehicles involved in an accident?
A. less than 11%
B. at least 11% but less than 13%
C. at least 13% but less than 15%
D. at least 15% but less than 17%
E. at least 17%

13.5 (2 points) What is the median number of vehicles involved in an accident?
A. 1  B. 2  C. 3  D. 4  E. 5
Use the following information for the next four questions:
The number of family members is given by a Zero-Truncated Negative Binomial Distribution with parameters $r = 4$ and $\beta = 0.5$.

13.6 (1 point) What is the mean number of family members?
A. less than 2.0
B. at least 2.0 but less than 2.1
C. at least 2.1 but less than 2.2
D. at least 2.2 but less than 2.3
E. at least 2.3

13.7 (2 points) What is the variance of the number of family members?
A. less than 2.0
B. at least 2.0 but less than 2.2
C. at least 2.2 but less than 2.4
D. at least 2.4 but less than 2.6
E. at least 2.6

13.8 (2 points) What is the chance of a family having 7 members?
A. less than 1.1%
B. at least 1.1% but less than 1.3%
C. at least 1.3% but less than 1.5%
D. at least 1.5% but less than 1.7%
E. at least 1.7%

13.9 (3 points) What is the probability of a family having more than 5 members?
A. less than 1%
B. at least 1%, but less than 3%
C. at least 3%, but less than 5%
D. at least 5%, but less than 7%
E. at least 7%
Use the following information for the next three questions:
A Logarithmic Distribution with parameter $\beta = 2$.

**13.10** (1 point) What is the mean?
A. less than 2.0  
B. at least 2.0 but less than 2.1  
C. at least 2.1 but less than 2.2  
D. at least 2.2 but less than 2.3  
E. at least 2.3

**13.11** (2 points) What is the variance?
A. less than 2.0  
B. at least 2.0 but less than 2.2  
C. at least 2.2 but less than 2.4  
D. at least 2.4 but less than 2.6  
E. at least 2.6

**13.12** (1 point) What is the density function at 6?
A. less than 1.1%  
B. at least 1.1% but less than 1.3%  
C. at least 1.3% but less than 1.5%  
D. at least 1.5% but less than 1.7%  
E. at least 1.7%

**13.13** (1 point) For a Zero-Truncated Negative Binomial Distribution with parameters $r = -0.6$ and $\beta = 3$, what is the density function at 5?
A. less than 1.1%  
B. at least 1.1% but less than 1.3%  
C. at least 1.3% but less than 1.5%  
D. at least 1.5% but less than 1.7%  
E. at least 1.7%
Use the following information for the next five questions:

The number of days per hospital stay is given by a Zero-Truncated Poisson Distribution with parameter \( \lambda = 2.5 \).

13.14 (1 point) What is the mean number of days per hospital stay?
A. less than 2.5
B. at least 2.5 but less than 2.6
C. at least 2.6 but less than 2.7
D. at least 2.7 but less than 2.8
E. at least 2.8

13.15 (2 points) What is the variance of the number of days per hospital stay?
A. less than 2.2
B. at least 2.2 but less than 2.3
C. at least 2.3 but less than 2.4
D. at least 2.4 but less than 2.5
E. at least 2.5

13.16 (1 point) What is the chance that a hospital stay is 6 days?
A. less than 3%
B. at least 3% but less than 4%
C. at least 4% but less than 5%
D. at least 5% but less than 6%
E. at least 6%

13.17 (2 points) What is the chance that a hospital stay is fewer than 4 days?
A. less than 50%
B. at least 50% but less than 60%
C. at least 60% but less than 70%
D. at least 70% but less than 80%
E. at least 80%

13.18 (2 points) What is the mode of this frequency distribution?
A. 1       B. 2       C. 3       D. 4       E. 5
Use the following information for the next 2 questions:

- Harvey Wallbanker, the Automatic Teller Machine, works 24 hours a day, seven days a week, without a vacation or even an occasional day off.
- Harvey services on average one customer every 10 minutes.
- 60% of Harvey’s customers are male and 40% are female.
- The gender of a customer is independent of the gender of the previous customers.
- Harvey’s hobby is to observe patterns of customers. For example, FMF denotes a female customer, followed by a male customer, followed by a female customer.

Harvey starts looking at customers who arrive after serving Pat, his most recent customer. How long does it take on average until he sees the following patterns.

13.19 (2 points) How long on average until Harvey sees “M”?

13.20 (2 points) How long on average until Harvey sees “F”?

13.21 (Course 151 Sample Exam #1, Q.12) (1.7 points)
A new business has initial capital 700 and will have annual net earnings of 1000. It faces the risk of a one time loss with the following characteristics:

- The loss occurs at the end of the year.
- The year of the loss is one plus a Geometric distribution with $\beta = .538$. (So the loss may either occur at the end of the first year, second year, etc.)
- The size of the loss is uniformly distributed on the ten integers: 500, 1000, 1500, ..., 5000.

Determine the probability of ruin.

(A) 0.00  (B) 0.41  (C) 0.46  (D) 0.60  (E) 0.65
Section 14, Zero-Modified Distributions

Frequency distributions can be constructed whose densities on the positive integers are proportional to those of a well-known distribution, but with $f(0)$ having any value between zero and one.

For example, let $h(x) = \frac{e^{-3} \cdot 3^x}{x!} \cdot \frac{1 - 0.25}{1 - e^{-3}}$, for $x = 1, 2, 3, \ldots$, and $h(0) = 0.25$.

Exercise: Verify that the sum of this density is in fact unity.

[The sum of the Poisson Distribution from 0 to $\infty$ is 1. $\sum_{i=0}^{\infty} e^{-3} \cdot 3^x / x! = 1$.]

Therefore, $\sum_{i=1}^{\infty} e^{-3} \cdot 3^x / x! = 1 - e^{-3}$.

$\Rightarrow \sum_{i=1}^{\infty} h(x) = (1 - 0.25)$. $\Rightarrow \sum_{i=0}^{\infty} h(x) = 1 - 0.25 + 0.25 = 1$.]

This is just an example of a Poisson Distribution Modified at Zero, with $\lambda = 3$ and 25% probability placed at zero.

For a Zero-Modified distribution, an arbitrary amount of probability has been placed at zero. In the example above it is 25%. Loss Models uses $p_0^M$ to denote this probability at zero. The remaining probability is spread out proportional to some well-known distribution such as the Poisson. In general if $f$ is a distribution on 0, 1, 2, 3, ..., and $0 < p_0^M < 1$,

then $h(0) = p_0^M$, $h(x) = f(x) \cdot \frac{1 - p_0^M}{1 - f(0)}$, for $x = 1, 2, 3$, ... is a distribution on 0, 1, 2, 3, ...

Exercise: For a Poisson Distribution Modified at Zero, with $\lambda = 3$ and 25% probability placed at zero, what are the densities at 0, 1, 2, 3, and 4?

[Solution: For example the density at 4 is: $(.75)(3^4)e^{-3}/(4!(1 - e^{-3})) = 0.133$.]

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>0.250</td>
<td>0.118</td>
<td>0.177</td>
<td>0.177</td>
<td>0.133</td>
</tr>
</tbody>
</table>

In the case of a Zero-Modified Distribution, there is no relationship assumed between the density at zero and the other densities, other than the fact that all of the densities sum to one.

See Section 6.7 and Appendix B.3.2 in Loss Models.
We have the following four cases:

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Zero-Modified Distribution, ( f(0) = p_0^M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binomial</td>
<td>( f(x) = \frac{m! q^x (1-q)^{m-x}}{x! (m-x)!} \frac{1}{1-(1-q)^m} ) ( x = 1, 2, 3,\ldots, m )</td>
</tr>
<tr>
<td>Poisson</td>
<td>( f(x) = \frac{(1-p_0^M) e^{-\lambda} \lambda^x}{x!} \frac{1}{1-e^{-\lambda}} ) ( x = 1, 2, 3,\ldots )</td>
</tr>
<tr>
<td>Negative Binomial(^6)</td>
<td>( f(x) = \frac{(1-p_0^M) x! (r+1)...(r+x-1)}{1-1/(1+\beta)^r} \frac{\beta^x}{(1+\beta)^x+r} ) ( x = 1, 2, 3,\ldots )</td>
</tr>
<tr>
<td>Logarithmic</td>
<td>( f(x) = \frac{(1-p_0^M) \left(\frac{\beta}{1+\beta}\right)^x}{x \ln(1+\beta)} ) ( x = 1, 2, 3,\ldots )</td>
</tr>
</tbody>
</table>

These four zero-modified distributions complete the \((a, b, 1)\) class of frequency distributions.\(^97\)

They each follow the formula: \( f(x)/f(x-1) = a + b/x \), for \( x \geq 2 \).

Note that if \( p_0^M = 0 \), one has \( f(0) = 0 \) and the zero-modified distribution reduces to a zero-truncated distribution. However, even though it might be useful to think of the zero-truncated distributions as a special case of the zero-modified distributions, \textit{Loss Models} restricts the term zero-modified to those cases where \( f(0) > 0 \).

**Moments:**

The moments of a zero-modified distribution \( h \) are given in terms of those of \( f \) by

\[
E_h[X^n] = (1 - p_0^M) \frac{E_f[X^n]}{1 - f(0)}. \quad \text{For example for the Zero-Truncated Poisson the mean is:}
\]

\[
(1 - p_0^M) \frac{\lambda}{1 - e^{-\lambda}}, \quad \text{while the second moment is:} \quad (1 - p_0^M) \frac{\lambda + \lambda^2}{1 - e^{-\lambda}}.
\]

\(^6\) The zero-modified version of the Negative Binomial is referred to by \textit{Loss Models} as the Zero-Modified Extended Truncated Negative Binomial.

\(^97\) See Table 6.4 and Appendix B.3 in \textit{Loss Models}. 
Exercise: For a Zero-modified Poisson with $\lambda = 3$ and 25% chance of zero claims, what is the mean?

[Solution: Let $f(x)$ be the untruncated Poisson, and $h(x)$ be the zero-modified distribution. Then $h(x) = \frac{0.75 f(x)}{1 - f(0)}$, $x > 0$. The mean of $h$ is:

$$\sum_{x=1}^{\infty} x h(x) = \sum_{x=1}^{\infty} \frac{0.75 x f(x)}{1 - f(0)} = 0.75 \sum_{x=0}^{\infty} \frac{x f(x)}{1 - f(0)} = 0.75 \frac{\text{mean of } f}{1 - f(0)} = 0.75 \frac{\lambda}{1 - e^{-\lambda}} = (0.75) \frac{3}{1 - e^{-3}} = (0.75)(3.157) = 2.368.$$

Comment: The term involving $x = 0$ would contribute nothing to the mean.]

Note that in order to get the moments of a zero-modified distribution, one could first compute the moment of the zero-truncated distribution and then multiply by $(1 - p_0^M)$. For example, the mean of a zero-truncated Poisson with $\lambda = 3$ is 3.157. Then the mean of the zero-modified with $\lambda = 3$ and 25% chance of zero claims has a mean of: $(3.157)(1 - 0.25) = 2.368$.

Exercise: For a Negative Binomial with $r = 0.7$ and $\beta = 3$ what is the second moment?

[Solution: The mean is $(0.7)(3) = 2.1$, the variance is $(0.7)(3)(1+3) = 8.4$, so the second moment is: $8.4 + 2.1^2 = 12.81$.]

Exercise: For a Zero-Truncated Negative Binomial with $r = 0.7$ and $\beta = 3$ what is the second moment?

[Solution: For a Negative Binomial with $r = .7$ and $\beta = 3$,

the density at zero is: $1/(1+\beta)^r = 4^{-0.7} = 0.379$, and the second moment is 12.81.

Thus the second moment of the zero-truncated distribution is: $12.81/(1 - 0.379) = 20.62$.]

Exercise: For a Zero-Modified Negative Binomial with $r = 0.7$ and $\beta = 3$, with a 15% chance of zero claims, what is the second moment?

[Solution: For a Zero-Truncated Negative Binomial with $r = .7$ and $\beta = 3$, the second moment is 20.62. Thus the second moment of the zero-modified distribution with a 15% chance of zero claims is: $20.62(1 - 0.15) = 17.52$.]
**Probability Generating Functions:**

The zero-modified distribution, can be thought of a mixture of a point mass of probability at zero and a zero-truncated distribution. The probability generating function of a mixture is the mixture of the probability generating functions. A point of probability at zero, has a probability generating function $E[z^n] = E[z^0] = 1$. Therefore, the Probability generating function, $P(z) = E[z^N]$, for a zero-modified distribution can be obtained from that for zero-truncated distribution:

$$P_M(z) = p_0^M + (1 - p_0^M) P_T(z)$$

where $P_M(z)$ is the p.g.f. for the zero-modified distribution and $P_T(z)$ is the p.g.f. for the zero-truncated distribution, and $p_0^M$ is the probability at zero for the zero-modified distribution.

Exercise: What is the Probability Generating Function for a Zero-Modified Poisson Distribution, with 30% probability placed at zero?

**Solution:** For the zero-truncated Poisson, $P_T(z) = \{e^{\lambda z} - 1\}/(e^{\lambda} - 1)$.

$$P_M(z) = p_0^M + (1 - p_0^M) P_T(z) = 0.3 + 0.7 \frac{e^{\lambda z} - 1}{e^{\lambda} - 1}$$

One can derive this relationship as follows:

Let $g(n)$ be the zero modified distribution and $h(n)$ be the zero-truncated distribution.

Then $g(0) = p_0^M$ and $g(n) = h(n) (1 - p_0^M)$ for $n > 0$.

$$P_M(z) = \sum_{n=0}^{\infty} z^n g(n) = p_0^M + \sum_{n=1}^{\infty} z^n (1 - p_0^M) h(n) = p_0^M + (1 - p_0^M) P_T(z).$$
Problems:

Use the following information for the next four questions:
The number of claims per year is given by a Zero-Modified Binomial Distribution with parameters $q = 0.3$ and $m = 5$, and with 15% probability of zero claims.

14.1 (1 point) What is the mean number of claims over the coming year?
A. less than 1.4
B. at least 1.4 but less than 1.5
C. at least 1.5 but less than 1.6
D. at least 1.6 but less than 1.7
E. at least 1.7

14.2 (2 points) What is the variance of the number of claims per year?
A. less than .98
B. at least .98 but less than 1.00
C. at least 1.00 but less than 1.02
D. at least 1.02 but less than 1.04
E. at least 1.04

14.3 (1 point) What is the chance of observing 3 claims over the coming year?
A. less than 13.0%
B. at least 13.0% but less than 13.4%
C. at least 13.4% but less than 13.8%
D. at least 13.8% but less than 14.2%
E. at least 14.2%

14.4 (2 points) What is the 95th percentile of the distribution of the number of claims per year?
A. 1   B. 2   C. 3   D. 4   E. 5
Use the following information for the next four questions:
The number of claims per year is given by a Zero-Modified Negative Binomial Distribution with parameters $r = 4$ and $\beta = 0.5$, and with 35\% chance of zero claims.

14.5 (1 point) What is the mean number of claims over the coming year?
A. less than 1.7
B. at least 1.7 but less than 1.8
C. at least 1.8 but less than 1.9
D. at least 1.9 but less than 2.0
E. at least 2.0

14.6 (2 points) What is the variance of the number of claims year?
A. less than 2.0
B. at least 2.0 but less than 2.2
C. at least 2.2 but less than 2.4
D. at least 2.4 but less than 2.6
E. at least 2.6

14.7 (1 point) What is the chance of observing 7 claims over the coming year?
A. less than .8\%
B. at least .8\% but less than 1.0\%
C. at least 1.0\% but less than 1.2\%
D. at least 1.2\% but less than 1.4\%
E. at least 1.4\%

14.8 (3 points) What is the probability of more than 5 claims in the coming year?
A. less than 1\%
B. at least 1\%, but less than 3\%
C. at least 3\%, but less than 5\%
D. at least 5\%, but less than 7\%
E. at least 7\%
Use the following information for the next three questions:

The number of claims per year is given by a Zero-Modified Logarithmic Distribution with parameter $\beta = 2$, and a 25% chance of zero claims.

14.9 (1 point) What is the mean number of claims over the coming year?
A. less than 1.0
B. at least 1.0 but less than 1.1
C. at least 1.1 but less than 1.2
D. at least 1.2 but less than 1.3
E. at least 1.3

14.10 (2 points) What is the variance of the number of claims per year?
A. less than 2.0
B. at least 2.0 but less than 2.2
C. at least 2.2 but less than 2.4
D. at least 2.4 but less than 2.6
E. at least 2.6

14.11 (1 point) What is the chance of observing 6 claims over the coming year?
A. less than 1.1%
B. at least 1.1% but less than 1.3%
C. at least 1.3% but less than 1.5%
D. at least 1.5% but less than 1.7%
E. at least 1.7%

14.12 (1 point) The number of claims per year is given by a Zero-Modified Negative Binomial Distribution with parameters $r = -.6$ and $\beta = 3$, and with a 20% chance of zero claims.
What is the chance of observing 5 claims over the coming year?
A. less than .8%
B. at least .8% but less than 1.0%
C. at least 1.0% but less than 1.2%
D. at least 1.2% but less than 1.4%
E. at least 1.4%
Use the following information for the next six questions:

The number of claims per year is given by a Zero-Modified Poisson Distribution with parameter $\lambda = 2.5$, and with 30% chance of zero claims.

14.13 (1 point) What is the mean number of claims over the coming year?
A. less than 2.0
B. at least 2.0 but less than 2.1
C. at least 2.1 but less than 2.2
D. at least 2.2 but less than 2.3
E. at least 2.3

14.14 (2 points) What is the variance of the number of claims per year?
A. less than 2.7
B. at least 2.7 but less than 2.8
C. at least 2.8 but less than 2.9
D. at least 2.9 but less than 3.0
E. at least 3.0

14.15 (1 point) What is the chance of observing 6 claims over the coming year?
A. less than 2%
B. at least 2% but less than 3%
C. at least 3% but less than 4%
D. at least 4% but less than 5%
E. at least 5%

14.16 (1 point) What is the chance of observing 2 claims over the coming year?
A. 18%  B. 20%  C. 22%  D. 24%  E. 26%

14.17 (2 points) What is the chance of observing fewer than 4 claims over the coming year?
A. less than 70%
B. at least 70% but less than 75%
C. at least 75% but less than 80%
D. at least 80% but less than 85%
E. at least 85%

14.18 (2 points) What is the mode of this frequency distribution?
A. 0  B. 1  C. 2  D. 3  E. 4
14.19 (3 points) Let \( p_k \) denotes the probability that the number of claims equals \( k \) for \( k = 0, 1, \ldots \). If \( p_n / p_m = 2.4^{n-m} m! / n! \), for \( m \geq 0, n \geq 0 \), then using the corresponding zero-modified claim count distribution with \( p_0^M = 0.31 \), calculate \( p_3^M \).

(A) 16%  (B) 18%  (C) 20%  (D) 22%  (E) 24%

14.20 (3 points) The following data is the number sick days taken at a large company during the previous year.

<table>
<thead>
<tr>
<th>Number of days:</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8+</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of employees:</td>
<td>50,122</td>
<td>9190</td>
<td>5509</td>
<td>3258</td>
<td>1944</td>
<td>1160</td>
<td>693</td>
<td>418</td>
<td>621</td>
</tr>
</tbody>
</table>

Is it likely that this data was drawn from a member of the \((a, b, 0)\) class?

Is it likely that this data was drawn from a member of the \((a, b, 1)\) class?

14.21 (3, 5/00, Q.37) (2.5 points) Given:

(i) \( p_k \) denotes the probability that the number of claims equals \( k \) for \( k = 0, 1, 2, \ldots \)

(ii) \( p_n / p_m = m! / n! \), for \( m \geq 0, n \geq 0 \)

Using the corresponding zero-modified claim count distribution with \( p_0^M = 0.1 \), calculate \( p_1^M \).

(A) 0.1  (B) 0.3  (C) 0.5  (D) 0.7  (E) 0.9
Section 15, Compound Frequency Distributions

A compound frequency distribution has a primary and secondary distribution, each of which is a frequency distribution. The primary distribution determines how many independent random draws from the secondary distribution we sum.

For example, assume the number of taxicabs that arrive per minute at the Heartbreak Hotel is Poisson with mean 1.3. In addition, assume that the number of passengers dropped off at the hotel by each taxicab is Binomial with q = 0.4 and m = 5. The number of passengers dropped off by each taxicab is independent of the number of taxicabs that arrive and is independent of the number of passengers dropped off by any other taxicab.

Then the aggregate number of passengers dropped off per minute at the Heartbreak Hotel is an example of a compound frequency distribution. It is a compound Poisson-Binomial distribution, with parameters $\lambda = 1.3$, $q = 0.4$, $m = 5$.

The distribution function of the primary Poisson is as follows:

<table>
<thead>
<tr>
<th>Number of Claims</th>
<th>Probability Density Function</th>
<th>Cumulative Distribution Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>27.253%</td>
<td>0.27253</td>
</tr>
<tr>
<td>1</td>
<td>35.429%</td>
<td>0.62682</td>
</tr>
<tr>
<td>2</td>
<td>23.029%</td>
<td>0.85711</td>
</tr>
<tr>
<td>3</td>
<td>9.979%</td>
<td>0.95690</td>
</tr>
<tr>
<td>4</td>
<td>3.243%</td>
<td>0.98934</td>
</tr>
<tr>
<td>5</td>
<td>0.843%</td>
<td>0.99777</td>
</tr>
<tr>
<td>6</td>
<td>0.183%</td>
<td>0.99960</td>
</tr>
</tbody>
</table>

So for example, there is a 3.243% chance that 4 taxicabs arrive; in which case the number passengers dropped off is the sum of 4 independent identically distributed Binomials, given by the secondary Binomial Distribution. There is a 27.253% chance there are no taxicabs, a 35.429% chance we take one Binomial, 23.029% chance we sum the result of 2 independent identically distributed Binomials, etc.

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98 See Section 6.8 of Loss Models, not on the syllabus. However, compound distributions are mathematically the same as aggregate distributions. See “Mahler’s Guide to Aggregate Distributions.” Some of you may better understand the idea of compound distributions by seeing how they are simulated in “Mahler’s Guide to Simulation.”

99 In the name of a compound distribution, the primary distribution is listed first and the secondary distribution is listed second.

100 While we happen to know that the sum of 4 independent Binomials each with q = .4, m = 5 is another Binomial with parameters q = .4, m = 20, that fact is not essential to the general concept of a compound distribution.
The secondary Binomial Distribution with \( q = 0.4, m = 5 \) is as follows:

<table>
<thead>
<tr>
<th>Number of Claims</th>
<th>Probability Density Function</th>
<th>Cumulative Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7.776%</td>
<td>0.07776</td>
</tr>
<tr>
<td>1</td>
<td>25.920%</td>
<td>0.33696</td>
</tr>
<tr>
<td>2</td>
<td>34.560%</td>
<td>0.68256</td>
</tr>
<tr>
<td>3</td>
<td>23.040%</td>
<td>0.91296</td>
</tr>
<tr>
<td>4</td>
<td>7.680%</td>
<td>0.98976</td>
</tr>
<tr>
<td>5</td>
<td>1.024%</td>
<td>1.00000</td>
</tr>
</tbody>
</table>

Thus assuming a taxicab arrives, there is a 34.560% chance that 2 passengers are dropped off.

In this example, the primary distribution determines how many taxicabs arrive, while the secondary distribution determines the number of passengers departing per taxicab. Instead, the primary distribution could be the number of envelopes arriving and the secondary distribution could be the number of claims in each envelope.\(^{101}\)

**Actuaries often use compound distributions when the primary distribution determines how many accidents there are, while for each accident the number of persons injured or number of claimants is determined by the secondary distribution.**\(^{102}\) This particular model, while useful for comprehension, may or may not apply to any particular use of the mathematical concept of compound frequency distributions.

There are number of methods of computing the density of compound distributions, among them the use of convolutions and the use of the Recursive Method (Panjer Algorithm.)\(^ {103}\)

**Probability Generating Function of Compound Distributions:**

One can get the Probability Generating Function of a compound distribution in terms of those of its primary and secondary distributions:

\[
P(z) = P_1[P_2(z)].
\]

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\(^{101}\) See 3, 11/01, Q.30.

\(^{102}\) See 3, 5/01, Q.36.

\(^{103}\) Both discussed in “Mahler’s Guide to Aggregate Distributions,” where they are applied to both compound and aggregate distributions.
Exercise: What is the Probability Generating Function of a Compound Geometric-Binomial Distribution, with $\beta = 3$, $q = 0.1$, and $m = 2$.

[Solution: The p.g.f. of the primary Geometric is: $1/(1 - 3(z-1)) = 1/(4 - 3z)$.

The p.g.f. of the secondary Binomial is: $(1 + (.1)(z-1))^2 = (.9 + .1z)^2 = .01z^2 + .18z + .81$.

$P(z) = P_1[P_2(z)] = 1/(4 - 3(.01z^2 + .18z + .81)) = -1/(.03z^2 + .54z - 1.57)$.

Recall, that for any frequency distribution, $f(0) = P(0)$. Therefore, for a compound distribution, $c(0) = P_c(0) = P_1[P_2(0)] = P_1[s(0)]$.

**compound density at 0 = p.g.f. of the primary at density at 0 of the secondary.**

For example, in the previous exercise, the density of the compound distribution at zero is its p.g.f. at $z = 0$: $1/1.57 = .637$. The density at 0 of the secondary Binomial Distribution is: $.9^2 = .81$. The p.g.f. of the primary distribution at .81 is: $1/(4 - (3)(.81)) = 1/1.57 = .637$.

If one takes the p.g.f. of a compound distribution to a power $\rho > 0$, $P(z)^{\rho} = P_1^{\rho}[P_2(z)]$.

Thus if the primary distribution is infinitely divisible, i.e., $P_1^{\rho}$ has the same form as $P_1$, then $P^{\rho}$ has the same form as $P$. If the primary distribution is infinitely divisible, then so is the compound distribution.

Since the Poisson and the Negative Binomial are each infinitely divisible, so are compound distributions with a primary distribution which is either a Poisson or a Negative Binomial (including a Geometric.)

**Adding Compound Distributions:**

For example, let us assume that taxi cabs arrive at a hotel (primary distribution) and drop people off (secondary distribution.) Assume two independent Compound Poisson Distributions with the same secondary distribution. The first compound distribution represents those cabs whose drivers were born in January through June and has $\lambda = 11$, while the second compound distribution represents those cabs whose drivers were born in July through December and has $\lambda = 9$.

Then the sum of the two distributions represents the passengers from all of the cabs, and is a Compound Poisson Distribution with $\lambda = 11 + 9 = 20$, and the same secondary distribution as each of the individual Compound Distributions.

Note that the parameter of the primary rather than secondary distribution was affected.

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104 This is the first step of the Panjer Algorithm, discussed in “Mahler’s Guide to Aggregate Distributions.”
Exercise: Let X be a Poisson-Binomial Distribution compound frequency distribution with $\lambda = 4.3$, $q = 0.2$, and $m = 5$. Let Y be a Poisson-Binomial Distribution compound frequency distribution with $\lambda = 2.4$, $q = 0.2$, and $m = 5$. What is the distribution of $X + Y$?
[Solution: A Poisson-Binomial Distribution with $\lambda = 4.3 + 2.4 = 6.7$, $q = 0.2$, and $m = 5$.]

The sum of two independent identically distributed Compound Poisson variables has the same form. The sum of two independent identically distributed Compound Negative Binomial variables has the same form.

Exercise: Let X be a Negative Binomial-Poisson compound frequency distribution with $\beta = 0.7$, $r = 2.5$, and $\lambda = 3$.
What is the distributional form of the sum of two independent random draws from X?
[Solution: A Negative Binomial-Poisson with $\beta = 0.7$, $r = (2)(2.5) = 5$, and $\lambda = 3$.]

Exercise: Let X be a Poisson-Geometric compound frequency distribution with $\lambda = 0.3$ and $\beta = 1.5$.
What is the distributional form of the sum of twenty independent random draws from X?
[Solution: The sum of 20 independent identically distributed variables is of the same form. However, $\lambda = (20)(0.3) = 6$. We get a Poisson-Geometric compound frequency distribution with $\lambda = 6$ and $\beta = 1.5$.]

If one adds independent identically distributed Compound Binomial variables one gets the same form.

Exercise: Let X be a Binomial-Geometric compound frequency distribution with $q = 0.2$, $m = 3$, and $\beta = 1.5$.
What is the distributional form of the sum of twenty independent random draws from X?
[Solution: The sum of 20 independent identically distributed binomial variables is of the same form, with $m = (20)(3) = 60$. We get a Binomial-Geometric compound frequency distribution with $q = .2$, $m = 60$, and $\beta = 1.5$.]

**Thinning Compound Distributions:**

Thinning compound distributions can be done in two different manners, one manner affects the primary distribution, and the other manner affects the secondary distribution.

For example, assume that taxi cabs arrive at a hotel (primary distribution) and drop people off (secondary distribution.) Then we can either select certain types of cabs or certain types of people. Depending on which we select, we affect the primary or secondary distribution.
Assume we select only those cabs that are less than one year old (and assume age of cab is independent of the number of people dropped off and the frequency of arrival of cabs.) Then this would affect the primary distribution, the number of cabs.

Exercise: Cabs arrive via a Poisson with mean 1.3. The number of people dropped off by each cab is Binomial with q = 0.2 and m = 5. The number of people dropped off per cab is independent of the number of cabs that arrive. 30% of cabs are less than a year old. The age of cabs is independent of the number of people dropped off and the frequency of arrival of cabs.
What is the distribution of the number of people dropped off by cabs less than one year old?
[Solution: Cabs less than a year old arrive via a Poisson with $\lambda = (30\%)(1.3) = .39$. There is no effect on the number of people per cab (secondary distribution.) We get a Poisson-Binomial Distribution compound frequency distribution with $\lambda = .39$, q = 0.2, and m = 5.]

This first manner of thinning affects the primary distribution. For example, it might occur if the primary distribution represents the number of accidents and the secondary distribution represents the number of claims.

For example, assume that the number of accidents is Negative Binomial with $\beta = 2$ and r = 30, and the number of claims per accident is Binomial with q = .3 and m = 7. Then the total number of claims is Compound Negative Binomial-Binomial with parameters $\beta = 2$, r = 30, q = .3 and m = 7.

Exercise: Accidents are assigned at random to one of four claims adjusters: Jerry, George, Elaine, or Cosmo.
What is the distribution of the number claims adjusted by George?
[Solution: We are selecting at random 1/4 of the accidents. We are thinning the Negative Binomial Distribution of the number of accidents. Therefore, the number of accidents assigned to George is Negative Binomial with $\beta = 2/4 = 0.5$ and r = 30.
The number claims adjusted by George is Compound Negative Binomial-Binomial with parameters $\beta = 0.5$, r = 30, q = .3 and m = 7.]

Returning to the cab example, assume we select only female passengers, (and gender of passenger is independent of the number of people dropped off and the frequency of arrival of cabs.). Then this would affect the secondary distribution, the number of passengers.
Exercise: Cabs arrive via a Poisson with mean 1.3. The number of people dropped off by each cab is Binomial with $q = 0.2$ and $m = 5$. The number of people dropped off per cab is independent of the number of cabs that arrive. 40% of the passengers are female. The gender of passengers is independent of the number of people dropped off and the frequency of arrival of cabs.

What is the distribution of the number of females dropped off by cabs?

[Solution: The distribution of female passengers per cab is Binomial with $q = (.4)(.2) = .08$ and $m = 5$. There is no effect on the number of cabs (primary distribution.) We get a Poisson-Binomial Distribution compound frequency distribution with $\lambda = 1.3$, $q = 0.08$, and $m = 5$.]

This second manner of thinning a compound distribution affects the secondary distribution. It is mathematically the same as what happens when one takes only the large claims in a frequency and severity situation, when the frequency distribution itself is compound.\(^{105}\)

For example, if frequency is Poisson-Binomial with $\lambda = 1.3$, $q = 0.2$, and $m = 5$, and 40% of the claims are large. The number of large claims would be simulated by first getting a random draw from the Poisson, then simulating the appropriate number of random Binomials, and then for each claim from the Binomial there is a 40% chance of selecting it at random independent of any other claims. This is mathematically the same as thinning the Binomial. Therefore, large claims have a Poisson-Binomial Distribution compound frequency distribution with $\lambda = 1.3$, $q = (.4)(.2) = 0.08$ and $m = 5$.

Exercise: Let frequency be given by a Geometric-Binomial compound frequency distribution with $\beta = 1.5$, $q = 0.2$, and $m = 3$. Severity follows an Exponential Distribution with mean 1000. Frequency and severity are independent.

What is the frequency distribution of losses of size between 500 and 2000?

[Solution: The fraction of losses that are of size between 500 and 2000 is:

$F(2000) - F(500) = (1-e^{-2000/1000}) - (1-e^{-500/1000}) = e^{-5} - e^{-2} = .4712$. Thus the losses of size between 500 and 2000 follow a Geometric-Binomial compound frequency distribution with $\beta = 1.5$, $q = (.4712)(.2) = .0942$, and $m = 3$.]

Proof of Some Thinning Results:\(^{106}\)

One can use the result for the probability generating function for a compound distribution, p.g.f. of compound distribution = p.g.f. of primary distribution[p.g.f. of secondary distribution], in order to determine the results of thinning a Poisson, Binomial, or Negative Binomial Distribution.

\(^{105}\) This is what is considered in Section 8.6 of Loss Models.

\(^{106}\) See Section 8.6 of Loss Models.
Assume one has a Poisson Distribution with mean $\lambda$.
Assume one selects at random 30% of the claims.
This is mathematically the same as a compound distribution with a primary distribution that is
Poisson with mean $\lambda$ and a secondary distribution that is Bernoulli with $q = .3$.

The p.g.f. of the Poisson is $P(z) = e^{\lambda(z-1)}$.
The p.g.f. of the Bernoulli is $P(z) = 1 + .3(z-1)$.
The p.g.f. of the compound distribution is obtained by replacing $z$ in the p.g.f. of the primary
Poisson with the p.g.f of the secondary Bernoulli:
$P(z) = \exp[\lambda(1 + .3(z-1) - 1)] = \exp[(.3\lambda)(z - 1)]$.

This is the p.g.f. of a Poisson Distribution with mean $.3\lambda$.
Thus the thinned distribution is also Poisson, with mean $.3\lambda$.

In general, when thinning a Poisson by a factor $t$, the thinned distribution is also Poisson with mean $t\lambda$.

Similarly, assume we are thinning a Binomial Distribution with parameters $q$ and $m$.
The p.g.f. of the Binomial is $P(z) = (1 + q(z-1))^m$.
This is mathematically the same as a compound distribution with secondary distribution a Bernoulli
with mean $t$.
The p.g.f. of this compound distribution is: $\{1 + q(1 + t(z-1) - 1)\}^m = \{1 + tq(z-1)\}^m$.
This is the p.g.f. of a Binomial Distribution with parameters $tq$ and $m$.

In general, when thinning a Binomial by a factor $t$, the thinned distribution is also Binomial with
parameters $tq$ and $m$.

Assume we are thinning a Negative Binomial Distribution with parameters $\beta$ and $r$.
The p.g.f. of the Negative Binomial is $P(z) = (1 - \beta(z-1))^r$.
This is mathematically the same as a compound distribution with secondary distribution a Bernoulli
with mean $t$.
The p.g.f. of this compound Distribution is: $\{1 - \beta(1 + t(z-1) - 1)\}^r = \{1 - t\beta((z-1))^r\}$.
This is the p.g.f. of a Negative Binomial Distribution with parameters $r$ and $t\beta$.
In general, when thinning a Negative Binomial by a factor $t$, the thinned distribution is also Negative
Binomial with parameters $t\beta$ and $r$. 
Problems:

15.1 (2 points) The number of accidents is Geometric with $\beta = 1.7$.
The number of claims per accident is Poisson with $\lambda = 3.1$.
For the total number of claims, what is the Probability Generating Function, $P(z)$?
A. $\exp[3.1(z - 1)]/(2.7 - 1.7z)$
B. $1/(2.7 - 1.7\exp[3.1(z - 1)])$
C. $\exp[3.1(z - 1)] + (2.7 - 1.7z)$
D. $\exp[3.1(z - 1.7)/(2.7 - 1.7z)]$
E. None of the above

15.2 (1 point) Frequency is given by a Poisson-Binomial compound frequency distribution, with $\lambda = .18$, $q = 0.3$, and $m = 3$.
One third of all losses are greater than $10,000$. Frequency and severity are independent.
What is frequency distribution of losses of size greater than $10,000$?
A. Compound Poisson-Binomial with $\lambda = .18$, $q = 0.3$, and $m = 3$.
B. Compound Poisson-Binomial with $\lambda = .18$, $q = 0.1$, and $m = 3$.
C. Compound Poisson-Binomial with $\lambda = .18$, $q = 0.3$, and $m = 1$.
D. Compound Poisson-Binomial with $\lambda = .06$, $q = 0.3$, and $m = 3$.
E. None of the above.

15.3 (1 point) $X$ is given by a Binomial-Geometric compound frequency distribution, with $q = .15$, $m = 3$, and $\beta = 2.3$. $Y$ is given by a Binomial-Geometric compound frequency distribution, with $q = .15$, $m = 5$, and $\beta = 2.3$. $X$ and $Y$ are independent.
What is the distributional form of $X + Y$?
A. Compound Binomial-Geometric with $q = .15$, $m = 4$, and $\beta = 2.3$
B. Compound Binomial-Geometric with $q = .15$, $m = 8$, and $\beta = 2.3$
C. Compound Binomial-Geometric with $q = .15$, $m = 4$, and $\beta = 4.6$
D. Compound Binomial-Geometric with $q = .15$, $m = 8$, and $\beta = 4.6$
E. None of the above.

15.4 (2 points) A compound claims frequency model has the following properties:
(i) The primary distribution has probability generating function:
$$P(z) = .2z + .5z^2 + .3z^3.$$  
(ii) The secondary distribution has probability generating function:
$$P(z) = \exp[.7(z - 1)].$$
Calculate the probability of no claims from this compound distribution.
(A) 18%  (B) 20%  (C) 22%  (D) 24%  (E) 26%
15.5 (1 point) Assume each exposure has a Poisson-Poisson compound frequency distribution, as per Loss Models, with $\lambda_1 = .03$ and $\lambda_2 = .07$. You insure 20,000 independent exposures. What is the frequency distribution for your portfolio?
A. Compound Poisson-Poisson with $\lambda_1 = .03$ and $\lambda_2 = .07$
B. Compound Poisson-Poisson with $\lambda_1 = .03$ and $\lambda_2 = 1400$
C. Compound Poisson-Poisson with $\lambda_1 = 600$ and $\lambda_2 = .07$
D. Compound Poisson-Poisson with $\lambda_1 = 600$ and $\lambda_2 = 1400$
E. None of the above.

15.6 (2 points) Frequency is given by a Poisson-Binomial compound frequency distribution, with parameters $\lambda = 1.2$, $q = 0.1$, and $m = 4$. What is the Probability Generating Function?
A. $\{1 + .1(z - 1)\}^4$
B. $\exp(1.2(z - 1))$
C. $\exp[1.2((1 + .1(z - 1))\}^4 - 1)]$
D. $\{1 + .1(\exp[1.2(z - 1)] - 1))\}^4$
E. None of the above.

15.7 (1 point) The total number of claims from a book of business with 100 exposures has a Compound Poisson-Geometric Distribution with $\lambda = 4$ and $\beta = 0.8$. Next year this book of business will have 75 exposures. Next year, what is the distribution of the total number of claims from this book of business?
A. Compound Poisson-Geometric with $\lambda = 4$ and $\beta = 0.8$
B. Compound Poisson-Geometric with $\lambda = 3$ and $\beta = 0.8$
C. Compound Poisson-Geometric with $\lambda = 4$ and $\beta = 0.6$
D. Compound Poisson-Geometric with $\lambda = 3$ and $\beta = 0.6$
E. None of the above.

15.8 (2 points) A compound claims frequency model has the following properties:
(i) The primary distribution has probability generating function:
$$P(z) = 1/(5 - 4z).$$
(ii) The secondary distribution has probability generating function:
$$P(z) = (.8 + .2z)^3.$$
Calculate the probability of no claims from this compound distribution.
(A) 28% (B) 30% (C) 32% (D) 34% (E) 36%
15.9 (1 point) The total number of claims from a group of 50 drivers has a Compound Negative Binomial-Poisson Distribution with $\beta = 0.4$, $r = 3$, and $\lambda = 0.7$. What is the distribution of the total number of claims from 500 similar drivers?
A. Compound Negative Binomial-Poisson with $\beta = 0.4$, $r = 30$, and $\lambda = 0.7$.
B. Compound Negative Binomial-Poisson with $\beta = 4$, $r = 3$, and $\lambda = 0.7$.
C. Compound Negative Binomial-Poisson with $\beta = 0.4$, $r = 3$, and $\lambda = 7$.
D. Compound Negative Binomial-Poisson with $\beta = 4$, $r = 30$, and $\lambda = 7$.
E. None of the above.

15.10 (SOA M, 11/05, Q.27) (2.5 points)
An actuary has created a compound claims frequency model with the following properties:
(i) The primary distribution is the negative binomial with probability generating function
$$P(z) = [1 - 3(z - 1)]^{-2}.$$ 
(ii) The secondary distribution is the Poisson with probability generating function
$$P(z) = \exp[\lambda(z - 1)].$$
(iii) The probability of no claims equals 0.067.
Calculate $\lambda$.
(A) 0.1    (B) 0.4    (C) 1.6    (D) 2.7    (E) 3.1
Section 16, Moments of Compound Frequency Distributions\(^{107}\)

One may find it helpful to think of the secondary distribution as taking the role of a severity distribution in the calculation of aggregate losses.\(^{108}\) Since the situations are mathematically equivalent, many of the techniques and formulas that apply to aggregate losses apply to compound frequency distributions.

For example, the same formulas for the mean, variance and skewness apply.\(^{109}\)

**Mean of Compound Dist.**
\[
\text{Mean of Compound Dist.} = \text{(Mean of Primary Dist.) (Mean of Secondary Dist.)}
\]

**Variance of Compound Dist.**
\[
\text{Variance of Compound Dist.} = \text{(Mean of Primary Dist.) (Variance of Secondary Dist.) + (Mean of Secondary Dist.)}^2 \text{ (Variance of Primary Dist.)}
\]

**Skewness Compound Dist.**
\[
\text{Skewness Compound Dist.} = \frac{\{(\text{Mean of Primary Dist.) (Variance of Secondary Dist.)}^{3/2} (\text{Skewness of Secondary Dist.)} + 3(\text{Variance of Primary Dist.) (Mean of Secondary Dist.) (Variance of Secondary Dist.)} + (\text{Variance of Primary Dist.)}^{3/2} (\text{Skewness of Primary Dist.) (Mean of Secondary Dist.)}^3 \}}{\text{(Variance of Compound Dist.)}^{3/2}}
\]

Exercise: What are the mean and variance of a compound Poisson-Binomial distribution, with parameters \(\lambda = 1.3, q = .4, m = 5\).

[Solution: The mean and variance of the primary Poisson Distribution are both 1.3. The mean and variance of the secondary Binomial Distribution are (.4)(5) = 2 and (.4)(.6)(5) = 1.2. Thus the mean of the compound distribution is (1.3)(2) = 2.6. The variance of the compound distribution is (1.3)(1.2) + (2)^2(1.3) = 6.76.]

\(^{107}\) See Section 6.8 of *Loss Models*, not on the syllabus. However, since compound distributions are mathematically the same as aggregate distributions, I believe that a majority of the questions in this section would be legitimate questions for your exam. Compound frequency distributions used to be on the syllabus.

\(^{108}\) In the case of aggregate losses, the frequency distribution determines how many independent identically distributed severity variables we will sum.

\(^{109}\) The secondary distribution takes the place of the severity, while the primary distribution takes the place of the frequency, in the formulas involving aggregate losses. \(\sigma_{agg}^2 = \mu_F \sigma_S^2 + \mu_S^2 \sigma_F^2\).

See “Mahler’s Guide to Aggregate Distributions.”
Thus in the case of the Heartbreak Hotel example in the previous section,\textsuperscript{110} on average 2.6 passengers are dropped off per minute. The variance of the number of passengers dropped off is 6.76.

**Poisson Primary Distribution:**

In the case of a Poisson primary distribution with mean $\lambda$, the variance of the compound distribution could be rewritten as:

\[
\lambda(\text{Variance of Secondary Dist.}) + (\text{Mean of Secondary Dist.})^2 = \lambda(\text{Variance of Secondary Dist. + Mean of Secondary Dist.}^2 ) = \lambda(2\text{nd moment of Secondary Distribution}).
\]

It also turns out that the third central moment of a compound Poisson distribution $= \lambda(3\text{rd moment of Secondary Distribution}).$

For a Compound Poisson Distribution:

**Mean** $= \lambda(\text{mean of Secondary Distribution}).$

**Variance** $= \lambda(2\text{nd moment of Secondary Distribution}).$

**3rd central moment** $= \lambda(3\text{rd moment of Secondary Distribution}).$

**Skewness** $= \lambda^{-1.5}(3\text{rd moment of Second. Dist.})/(2\text{nd moment of Second. Dist.})^{1.5}$.\textsuperscript{111}

**Exercise:** The number of accidents follows a Poisson Distribution with $\lambda = .04$. Each accident generates 1, 2 or 3 claimants with probabilities 60%, 30%, and 10%. Determine the mean, variance, and skewness of the total number of claimants.

[Solution: The secondary distribution has mean 1.5, second moment 2.7, and third moment 5.7. Thus the mean number of claimants is: $(.04)(1.5) = .06$. The variance of the number of claimants is: $(.04)(2.7) = .108$. The skewness of the number of claimants is: $(.04^{-1.5})(5.7)/(2.7)^{1.5} = 6.42].$

\textsuperscript{110} The number of taxicabs that arrive per minute at the Heartbreak Hotel is Poisson with mean 1.3. The number of passengers dropped off at the hotel by each taxicab is Binomial with $q = .4$ and $m = 5$. The number of passengers dropped off by each taxicab is independent of the number of taxicabs that arrive and is independent of the number of passengers dropped off by any other taxicab.

\textsuperscript{111} Skewness $= (\text{third central moment})/ \text{Variance}^{1.5}$. 
Problems:

16.1 (1 point) For a compound distribution:
Mean of primary distribution = 15.
Standard Deviation of primary distribution = 3.
Mean of secondary distribution = 10.
Standard Deviation of secondary distribution = 4.
What is the standard deviation of the compound distribution?
A. 26  B. 28  C. 30  D. 32  E. 34

16.2 (2 points) The number of accidents follows a Poisson distribution with mean 10 per month.
Each accident generates 1, 2, or 3 claimants with probabilities 40%, 40%, 20%, respectively.
Calculate the variance in the total number of claimants in a year.
A. 250  B. 300  C. 350  D. 400  E. 450

Use the following information for the next 3 questions:
The number of customers per minute is Geometric with $\beta = 1.7$.
The number of items sold to each customer is Poisson with $\lambda = 3.1$.
The number of items sold per customer is independent of the number of customers.

16.3 (1 point) What is the mean?
A. less than 5.0  B. at least 5.0 but less than 5.5  C. at least 5.5 but less than 6.0
D. at least 6.0 but less than 6.5  E. at least 6.5

16.4 (1 point) What is the variance?
A. less than 50  B. at least 50 but less than 51  C. at least 51 but less than 52
D. at least 52 but less than 53  E. at least 53

16.5 (2 points) What is the chance that more than 4 items are sold during the next minute?
Use the Normal Approximation.
A. 46%  B. 48%  C. 50%  D. 52%  E. 54%
16.6 (3 points) A dam is proposed for a river which is currently used for salmon breeding. You have modeled:
(i) For each hour the dam is opened the number of female salmon that will pass through and reach the breeding grounds has a distribution with mean 50 and variance 100.
(ii) The number of eggs released by each female salmon has a distribution with mean of 3000 and variance of 1 million.
(iii) The number of female salmon going through the dam each hour it is open and the numbers of eggs released by the female salmon are independent.
Using the normal approximation for the aggregate number of eggs released, determine the least number of whole hours the dam should be left open so the probability that 2 million eggs will be released is greater than 99.5%.
(A) 14 (B) 15 (C) 16 (D) 17 (E) 18

16.7 (3 points) The claims department of an insurance company receives envelopes with claims for insurance coverage at a Poisson rate of \( \lambda = 7 \) envelopes per day. For any period of time, the number of envelopes and the numbers of claims in the envelopes are independent. The numbers of claims in the envelopes have the following distribution:

<table>
<thead>
<tr>
<th>Number of Claims</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.60</td>
</tr>
<tr>
<td>2</td>
<td>0.30</td>
</tr>
<tr>
<td>3</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Using the normal approximation, calculate the 99\(^{th}\) percentile of the number of claims received in 5 days.
(A) 73 (B) 75 (C) 77 (D) 79 (E) 81

16.8 (3 points) The number of persons using an ATM per hour has a Negative Binomial Distribution with \( \beta = 2 \) and \( r = 13 \). Each hour is independent of the others.
The number of transactions per person has the following distribution:

<table>
<thead>
<tr>
<th>Number of Transactions</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.30</td>
</tr>
<tr>
<td>2</td>
<td>0.40</td>
</tr>
<tr>
<td>3</td>
<td>0.20</td>
</tr>
<tr>
<td>4</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Using the normal approximation, calculate the 80\(^{th}\) percentile of the number of transactions in 5 hours.
A. 300 B. 305 C. 310 D. 315 E. 320
Use the following information for the next 3 questions:

• The number of automobile accidents follows a Negative Binomial distribution with $\beta = 0.6$ and $r = 100$.
• For each automobile accident the number of claimants with bodily injury follows a Binomial Distribution with $q = 0.1$ and $m = 4$.
• The number of claimants with bodily injury is independent between accidents.

16.9 (2 points) Calculate the variance in the total number of claimants.

(A) 33 (B) 34 (C) 35 (D) 36 (E) 37

16.10 (1 point) What is probability that there are 20 or fewer claimants in total?

(A) 22% (B) 24% (C) 26% (D) 28% (E) 30%

16.11 (3 points) The amount of the payment to each claimant follows a Gamma Distribution with $\alpha = 3$ and $\theta = 4000$. The amount of payments to different claimants are independent of each other and are independent of the number of claimants. What is the probability that the aggregate payment exceeds 300,000?

(A) 44% (B) 46% (C) 48% (D) 50% (E) 52%

16.12 (3 points) The number of batters per half-inning of a baseball game is: $3 + \text{a Negative Binomial Distribution with } \beta = 1 \text{ and } r = 1.4$.
The number of pitches thrown per batter is: $1 + \text{a Negative Binomial Distribution with } \beta = 1.5 \text{ and } r = 1.8$.
What is the probability of more than 30 pitches in a half-inning?
Use the normal approximation with continuity correction.

(A) 1/2% (B) 1% (C) 2% (D) 3% (E) 4%

16.13 (3 points) The number of taxicabs that arrive per minute at the Gotham City Railroad Station is Poisson with mean 5.6. The number of passengers dropped off at the station by each taxicab is Binomial with $q = .3$ and $m = 4$. The number of passengers dropped off by each taxicab is independent of the number of taxicabs that arrive and is independent of the number of passengers dropped off by any other taxicab. Using the normal approximation for the aggregate passengers dropped off, determine the least number of whole minutes one must observe in order that the probability that at least 1000 passengers will be dropped off is greater than 90%.

(A) 155 (B) 156 (C) 157 (D) 158 (E) 159
16.14 (4 points) At a storefront legal clinic, the number of lawyers who volunteer to provide legal aid to the poor on any day is uniformly distributed on the integers 1 through 4. The number of hours each lawyer volunteers on a given day is Binomial with $q = 0.6$ and $m = 7$. The number of clients that can be served by a given lawyer per hour is a Poisson distribution with mean 5. Determine the probability that 40 or more clients can be served in a day at this storefront law clinic, using the normal approximation.

(A) 69%  (B) 71%  (C) 73%  (D) 75%  (E) 77%

Use the following information for the next 3 questions:
The number of persons entering a library per minute is Poisson with $\lambda = 1.2$.
The number of books returned per person is Binomial with $q = 0.1$ and $m = 4$.
The number of books returned per person is independent of the number of persons.

16.15 (1 point) What is the mean number of books returned per minute?
A. less than 0.5
B. at least 0.6 but less than 0.7
C. at least 0.7 but less than 0.8
D. at least 0.8 but less than 0.9
E. at least 0.9

16.16 (1 point) What is the variance of the number of books returned per minute?
A. less than 0.6
B. at least 0.6 but less than 0.7
C. at least 0.7 but less than 0.8
D. at least 0.8 but less than 0.9
E. at least 0.9

16.17 (1 point) What is the probability of observing more than two books returned in the next minute?
Use the Normal Approximation.
A. less than 0.6%
B. at least 0.6% but less than 0.7%
C. at least 0.7% but less than 0.8%
D. at least 0.8% but less than 0.9%
E. at least 0.9%
16.18 (2 points) Yosemite Sam is panning for gold. The number of pans with gold nuggets he finds per day is Poisson with mean 3. The number of nuggets per such pan are: 1, 5, or 25, with probabilities: 90%, 9%, and 1% respectively. The number of pans and the number of nuggets per pan are independent. Using the normal approximation with continuity correction, what is the probability that the number of nuggets found by Sam over the next ten days is less than 30?

(A) $\Phi(-1.2)$  (B) $\Phi(-1.1)$  (C) $\Phi(-1.0)$  (D) $\Phi(-0.9)$  (E) $\Phi(-0.8)$

16.19 (3, 11/00, Q.2) (2.5 points) In a clinic, physicians volunteer their time on a daily basis to provide care to those who are not eligible to obtain care otherwise. The number of physicians who volunteer in any day is uniformly distributed on the integers 1 through 5. The number of patients that can be served by a given physician has a Poisson distribution with mean 30. Determine the probability that 120 or more patients can be served in a day at the clinic, using the normal approximation with continuity correction.

(A) $1 - \Phi(0.68)$  (B) $1 - \Phi(0.72)$  (C) $1 - \Phi(0.93)$  (D) $1 - \Phi(3.13)$  (E) $1 - \Phi(3.16)$

16.20 (3, 5/01, Q.16) (2.5 points) A dam is proposed for a river which is currently used for salmon breeding. You have modeled:

(i) For each hour the dam is opened the number of salmon that will pass through and reach the breeding grounds has a distribution with mean 100 and variance 900.

(ii) The number of eggs released by each salmon has a distribution with mean of 5 and variance of 5.

(iii) The number of salmon going through the dam each hour it is open and the numbers of eggs released by the salmon are independent. Using the normal approximation for the aggregate number of eggs released, determine the least number of whole hours the dam should be left open so the probability that 10,000 eggs will be released is greater than 95%.

(A) 20  (B) 23  (C) 26  (D) 29  (E) 32

16.21 (3, 5/01, Q.36) (2.5 points) The number of accidents follows a Poisson distribution with mean 12. Each accident generates 1, 2, or 3 claimants with probabilities 1/2, 1/3, 1/6, respectively. Calculate the variance in the total number of claimants.

(A) 20  (B) 25  (C) 30  (D) 35  (E) 40
16.22 (3, 11/01, Q.30) (2.5 points) The claims department of an insurance company receives envelopes with claims for insurance coverage at a Poisson rate of $\lambda = 50$ envelopes per week. For any period of time, the number of envelopes and the numbers of claims in the envelopes are independent. The numbers of claims in the envelopes have the following distribution:

<table>
<thead>
<tr>
<th>Number of Claims</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.20</td>
</tr>
<tr>
<td>2</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>0.40</td>
</tr>
<tr>
<td>4</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Using the normal approximation, calculate the 90th percentile of the number of claims received in 13 weeks.

(A) 1690  (B) 1710  (C) 1730  (D) 1750  (E) 1770

16.23 (3, 11/02, Q.27) (2.5 points) At the beginning of each round of a game of chance the player pays 12.5. The player then rolls one die with outcome $N$. The player then rolls $N$ dice and wins an amount equal to the total of the numbers showing on the $N$ dice. All dice have 6 sides and are fair.

Using the normal approximation, calculate the probability that a player starting with 15,000 will have at least 15,000 after 1000 rounds.

(A) 0.01  (B) 0.04  (C) 0.06  (D) 0.09  (E) 0.12

16.24 (CAS3, 5/04, Q.26) (2.5 points) On Time Shuttle Service has one plane that travels from Appleton to Zebrashire and back each day. Flights are delayed at a Poisson rate of two per month. Each passenger on a delayed flight is compensated $100. The numbers of passengers on each flight are independent and distributed with mean 30 and standard deviation 50.

(You may assume that all months are 30 days long and that years are 360 days long.) Calculate the standard deviation of the annual compensation for delayed flights.

A. Less than $25,000
B. At least $25,000, but less than $50,000
C. At least $50,000, but less than $75,000
D. At least $75,000, but less than $100,000
E. At least $100,000
16.25 (SOA M, 11/05, Q.18) (2.5 points) In a CCRC, residents start each month in one of the following three states: Independent Living (State #1), Temporarily in a Health Center (State #2) or Permanently in a Health Center (State #3). Transitions between states occur at the end of the month. If a resident receives physical therapy, the number of sessions that the resident receives in a month has a geometric distribution with a mean which depends on the state in which the resident begins the month. The numbers of sessions received are independent. The number in each state at the beginning of a given month, the probability of needing physical therapy in the month, and the mean number of sessions received for residents receiving therapy are displayed in the following table:

<table>
<thead>
<tr>
<th>State#</th>
<th>Number in state</th>
<th>Probability of needing therapy</th>
<th>Mean number of visits</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>400</td>
<td>0.2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>300</td>
<td>0.5</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>200</td>
<td>0.3</td>
<td>9</td>
</tr>
</tbody>
</table>

Using the normal approximation for the aggregate distribution, calculate the probability that more than 3000 physical therapy sessions will be required for the given month.

(A) 0.21  (B) 0.27  (C) 0.34  (D) 0.42  (E) 0.50

16.26 (SOA M, 11/05, Q.39) (2.5 points) For an insurance portfolio:
(i) The number of claims has the probability distribution

\[
\begin{align*}
  n & \quad p_n \\
  0 & \quad 0.1 \\
  1 & \quad 0.4 \\
  2 & \quad 0.3 \\
  3 & \quad 0.2
\end{align*}
\]

(ii) Each claim amount has a Poisson distribution with mean 3; and
(iii) The number of claims and claim amounts are mutually independent.

Calculate the variance of aggregate claims.

(A) 4.8  (B) 6.4  (C) 8.0  (D) 10.2  (E) 12.4
16.27 (CAS3, 5/06, Q.35) (2.5 points)
The following information is known about a consumer electronics store:
- The number of people who make some type of purchase follows a Poisson distribution with a mean of 100 per day.
- The number of televisions bought by a purchasing customer follows a Negative Binomial distribution with parameters $r = 1.1$ and $\beta = 1.0$.

Using the normal approximation, calculate the minimum number of televisions the store must have in its inventory at the beginning of each day to ensure that the probability of its inventory being depleted during that day is no more than 1.0%.

A. Fewer than 138
B. At least 138, but fewer than 143
C. At least 143, but fewer than 148
D. At least 148, but fewer than 153
E. At least 153

16.28 (SOA M, 11/06, Q.30) (2.5 points)
You are the producer for the television show Actuarial Idol.
Each year, 1000 actuarial clubs audition for the show.
The probability of a club being accepted is 0.20.
The number of members of an accepted club has a distribution with mean 20 and variance 20.
Club acceptances and the numbers of club members are mutually independent.
Your annual budget for persons appearing on the show equals 10 times the expected number of persons plus 10 times the standard deviation of the number of persons.
Calculate your annual budget for persons appearing on the show.

(A) 42,600  (B) 44,200  (C) 45,800  (D) 47,400  (E) 49,000