Mahler’s Guide to
Frequency Distributions

Solutions to Problems
Sections 8-16

Joint Exam 4/C

prepared by
Howard C. Mahler, FCAS
Copyright ©2010 by Howard C. Mahler.

Study Aid F10-4-1H
Howard Mahler
hmahler@mac.com
Solutions to Problems, Sections 8 to 16

8.1. B. Variance = E[X^2] - E[X]^2 = 12.4 - 3.34^2 = 1.244.

Standard Deviation = √1.244 = 1.116.

skewness = \{E[X^3] - 3 E[X] E[X^2] + 2 E[X]^3\} / STDDEV^3 = 
\{48.82 - (3)(3.34)(12.4) + (2) (3.34^3)\} / (1.116^3) = -0.65.

<table>
<thead>
<tr>
<th>Number of Claims</th>
<th>Probability Density Function</th>
<th>Probability x # of Claims</th>
<th>Probability x Square of # of Claims</th>
<th>Probability x Cube of # of Claims</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2%</td>
<td>0</td>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>4%</td>
<td>0.04</td>
<td>0.04</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>14%</td>
<td>0.28</td>
<td>0.56</td>
<td>1.1</td>
</tr>
<tr>
<td>3</td>
<td>31%</td>
<td>0.93</td>
<td>2.79</td>
<td>8.4</td>
</tr>
<tr>
<td>4</td>
<td>36%</td>
<td>1.44</td>
<td>5.76</td>
<td>23.0</td>
</tr>
<tr>
<td>5</td>
<td>13%</td>
<td>0.65</td>
<td>3.25</td>
<td>16.2</td>
</tr>
<tr>
<td>Sum</td>
<td>1</td>
<td>3.34</td>
<td>12.4</td>
<td>48.82</td>
</tr>
</tbody>
</table>

8.2. D. \(\sigma^2 = \mu_2 - \mu_1^2 = (m + m^2) - m^2 = m.\)

skewness = \{\mu_3 - (3 \mu_1 \mu_2) + (2 \mu_1^3)\} / \sigma^3 = 
\{(m + 3m^2 + m^3) - 3(m + m^2)m + 2 m^3\} / m^{3/2} = m^{1.5}.

Comment: The moments are those of the Poisson Distribution with mean m.

8.3. B. E[X] = (7 + 3 + 5 + 10 + 5)/5 = 6. E[X^2] = (7^2 + 3^2 + 5^2 + 10^2 + 5^2)/5 = 41.6.

Var[X] = 41.6 - 6^2 = 5.6. E[X^3] = (7^3 + 3^3 + 5^3 + 10^3 + 5^3)/5 = 324.

Skewness = \{E[X^3] - 3 E[X^2]E[X] + 2E[X]^3\} / Var[X]^{1.5} 
= \{324 - (3)(41.6)(6) + (2)(6^3)\}/5.6^{1.5} = 7.2/13.25 = 0.54.

Comment: Similar to CAS3, 5/04, Q.28. E[(X-\bar{X})^3] = (1^3 + (-3)^3 + (-1)^3 + 4^3 + (-1)^3)/5 = 7.2.
8.4. E.  \( E[X] = \frac{12318}{100000} = 0.12318 \).  \( E[X^2] = \frac{14268}{100000} = 0.14268 \).

<table>
<thead>
<tr>
<th>Number of Claims</th>
<th>Number of Policies</th>
<th>Contribution to First Moment</th>
<th>Contribution to Second Moment</th>
<th>Contribution to Third Moment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>88,585</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>10,577</td>
<td>10577</td>
<td>10577</td>
<td>10577</td>
</tr>
<tr>
<td>2</td>
<td>779</td>
<td>1558</td>
<td>3116</td>
<td>6232</td>
</tr>
<tr>
<td>3</td>
<td>54</td>
<td>162</td>
<td>486</td>
<td>1458</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>16</td>
<td>64</td>
<td>256</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>5</td>
<td>25</td>
<td>125</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>100,000</strong></td>
<td><strong>12,318</strong></td>
<td><strong>14,268</strong></td>
<td><strong>18,648</strong></td>
</tr>
</tbody>
</table>

\( \text{Var}[X] = 0.14268 - 0.12318^2 = 0.12751 \).

\( \text{E}[X^3] = \frac{18648}{100000} = 0.18648 \).

Third Central Moment = \( \text{E}[X^3] - 3 \text{E}[X] \text{E}[X^2] + 2 \text{E}[X]^3 \)

\[ = 0.18648 - (3)(0.12318)(0.14268) + (2)(0.12318^3) = 0.13749. \]

Skewness = (Third Central Moment) / \( \text{Var}[X]^{1.5} = 0.13749/0.12751^{1.5} = 3.02 \).

**Comment:** Data taken from Table 5.9.1 in *Introductory Statistics with Applications in General Insurance* by Hossack, Pollard and Zehnwirth.

8.5. A. 1. True. The mean is \( \frac{(0)(800) + (1)(130) + (2)(50) + (3)(20)}{1000} = 0.290 \).

2. False. The second moment is \( \frac{(0^2)(800) + (1^2)(130) + (2^2)(50) + (3^2)(20)}{1000} = 0.510 \). Thus the variance = \( 0.510 - 0.29^2 = 0.4259 \).

3. False. The distribution is skewed to the right and thus of positive skewness. The third moment is:

\[ \frac{(0^3)(800) + (1^3)(130) + (2^3)(50) + (3^3)(20)}{1000} = 1.070. \]

Therefore, skewness = \( \frac{\{\mu_3 - (3 \mu_1 \mu_2) + (2 \mu_1^3)\}}{\text{STDDEV}^3} = \frac{1.070 - (3)(0.29)(0.51) + (2)(0.29^3)}{0.278} = 2.4 > 0 \).

8.6. E. \( E[X] = \frac{(4 + 1 + 3 + 2 + 15)}{5} = 5. \)  \( E[X^2] = \frac{(4^2 + 1^2 + 3^2 + 2^2 + 15^2)}{5} = 51. \)

\( \text{Var}[X] = 51 - 5^2 = 26. \)  \( E[X^3] = \frac{(4^3 + 1^3 + 3^3 + 2^3 + 15^3)}{5} = 695. \)

Skewness = \( \frac{\{E[X^3] - 3 E[X^2] E[X] + 2 E[X]^3\}}{\text{Var}[X]^{1.5}} \)

\[ = \frac{695 - (3)(51)(5) + (2)(5^3)}{26^{1.5}} = 180/133.425 = 1.358. \]

Alternatively, the third central moment is:

\( \frac{(4 - 5)^3 + (1 - 5)^3 + (3 - 5)^3 + (2 - 5)^3 + (15 - 5)^3}{5} = 180 \).  Skewness = \( 180/26^{1.5} = 1.358 \).

9.1. C.  \( P(z) = E[z^n] = (.35)(z^0) + (.25)(z^1) + (.20)(z^2) + (.15)(z^3) + (.05)(z^4) = 0.35 + 0.25z + 0.2z^2 + 0.15z^3 + 0.05z^4. \)
9.2. B. As shown in the Appendix B attached to the exam, for a Poisson Distribution 

\[ P(z) = e^{\lambda(z-1)} \]

\[ P(5) = e^{4\lambda} = e^{1.2} = 3.32 \]

9.3. A. The Binomial is not infinitely divisible. 
Comment: In the Binomial \( m \) is an integer. For \( m = 1 \) one has a Bernoulli. One can not divide a Bernoulli into smaller pieces.

9.4. C. \( P(z) = \frac{(e^{0.4z} - 1)}{(e^{0.4} - 1)} \). \( P'(z) = \frac{0.4e^{0.4z}}{(e^{0.4} - 1)} \). \( P''(z) = \frac{0.16e^{0.4z}}{(e^{0.4} - 1)} \). \( f(3) = \frac{(d^3 P(z) / dz^3)_{z=0}}{3!} = \frac{0.064/(e^{0.4} - 1))}{6} = 2.17\% \).
Comment: This is a zero-truncated Poisson Distribution with \( \lambda = 0.4 \), not on the syllabus.

9.5. D. The probability generating function of a sum of independent variables is the product of the probability generating functions.

\[ P_{B+C}(z) = P_B(z)P_C(z) = (.8z^2 + .2z^4)(.7z + .3z^5) = .56z^3 + .14z^5 + .24z^7 + .06z^9 \]

Alternately, B has 80% probability of being 2 and 20% probability of being 4. 
C has 70% probability of being 1 and 30% probability of being 5. 
Therefore, B + C has: (80%)(70%) = 56% chance of being 1 + 2 = 3, 
(20%)(70%) = 14% chance of being 4 + 1 = 5, (80%)(30%) = 24% chance of being 2 + 5 = 7, 
and (20%)(30%) = 6% chance of being 4 + 5 = 9. 
\[ \Rightarrow P_{B+C}(z) = (.8z^2 + .2z^4)(.7z + .3z^5) = .56z^3 + .14z^5 + .24z^7 + .06z^9 \]
Comment: An example of a convolution.

9.6. B. \( P(z) = \) Expected Value of \( z^n = \sum n^f(n) \). Thus \( f(2) = 0.3 \).

Alternately, \( P(z) = .5z + .3z^2 + .2z^4 \). \( P'(z) = .5 + .6z + .8z^3 \). \( P''(z) = .6 + .24z^3 \).
\[ f(2) = \frac{(d^2 P(z) / dz^2)_{z=0}}{2!} = .6/2 = 0.3 \]

9.7. E. As shown in the Appendix B attached to the exam, for a Binomial Distribution 

\[ P(z) = (1 + q(z-1))^m = (1 + (.7)(z-1))^4 \]

\[ P(10) = (1 + (.7)(9))^4 = 2840 \]

9.8. D. The p.g.f. of the Poisson Distribution is: \( P(z) = e^{\lambda(z-1)} = e^{5.6(z-1)} \).
\[ E[3^N] = P(3) = e^{5.6(3-1)} = e^{11.2} = 73,130 \].
9.9. D. For each Poisson, the probability generating function is: \( P(z) = \exp[\lambda_i(z-1)] \).

Multiplying a variable by a constant: \( P_{cX}[z] = E[z^{cX}] = E[(z^c)^X] = P_X[z^c] \).

For each Poisson times \( c \), the p.g.f. is: \( \exp[\lambda_i(z^c - 1)] \).

The p.g.f. of the sum of variables is a product of the p.g.f.s: \( P_Y(z) = \exp[(z^c - 1) \sum_{i=1}^{n} \lambda_i] \).

Comment: Multiplying a Poisson variable by a constant does not result in another Poisson; rather it results in what is called an Over-Dispersed Poisson Distribution. Since \( \text{Var}[cX]/E[cX] = c\text{Var}[X]/E[X] \), for a constant > 1, the Over-Dispersed Poisson Distribution has a variance greater than it mean. See for example “A Primer on the Exponential Family of Distributions”, by David R. Clark and Charles Thayer, CAS 2004 Discussion Paper Program.

9.10. (i) \( P'(z) = \{(1-\lambda t)(1 + \lambda t(1-z)) + \lambda t(z + \lambda t(1-z))\}/(1 + \lambda t(1-z))^2 = 1/(1 + \lambda t(1-z))^2 \).

\( E[X] = P'(1) = 1 \). The expected size of the population is 1 regardless of time. 
(ii) \( f(0) = P(0) = \lambda t/(1 + \lambda t) \). This is the probability of extinction by time \( t \).

The probability of survival to time \( t \) is: \( 1 - \lambda t/(1 + \lambda t) = 1/(1 + \lambda t) = (1/\lambda t)/(1/\lambda t + t) \), the survival function of a Pareto Distribution with \( \alpha = 1 \) and \( \theta = 1/\lambda t \).

(iii) As \( t \) approaches infinity, the probability of survival approaches zero.

Comment: \( P''(z) = 2\lambda t/(1 + \lambda t(1-z))^3 \). \( E[X(X - 1)] = P''(1) = 2\lambda t \).

\( \Rightarrow E[X^2] = E[X] + 2\lambda t = 1 + 2\lambda t. \Rightarrow \text{Var}[X] = 1 + 2\lambda t - 1^2 = 2\lambda t \).

9.11. \( P_1(z) = \exp[\mu_1(z-1)]. \) \( P_2(z) = \exp[\mu_2(z-1)]. \)

Since \( X_1 \) and \( X_2 \) are independent, the probability generating function of \( X_1 + X_2 \) is:

\( P_1(z)P_2(z) = \exp[\mu_1(z-1) + \mu_2(z-1)] = \exp[(\mu_1 + \mu_2)(z-1)] \).

This is the probability generating function of a Poisson with mean \( \mu_1 + \mu_2 \), which must therefore be the distribution of \( X_1 + X_2 \).

10.1. D. The 2nd factorial moment is: 
\( E[N(N-1)] = (.3)(0)(-1) + (.3)(1)(0) + (.2)(2)(1) + (.1)(3)(2) + (.1)(4)(3) = 2.2 \).
10.2. E. The factorial moments for a Poisson are: $\lambda^n$.

mean = first factorial moment = $\lambda = 5$.


$\Rightarrow E[X^3] = 125 + (3)(30) - (2)(5) = 205$.

Alternately, for the Poisson $P(z) = e^{\lambda(z-1)}$. 

$P(z) = e^{\lambda(z-1)}, \quad P'(z) = \lambda e^{\lambda(z-1)}, \quad P''(z) = \lambda^2 e^{\lambda(z-1)}$. 

mean = first factorial moment = $P'(1) = \lambda$.

Second factorial moment = $P''(1) = \lambda^2$.

Third factorial moment = $P'''(1) = \lambda^3$. Proceed as before.

Alternately, the skewness of a Poisson is $1/\sqrt{\lambda}$.

Since the variance is $\lambda$, the third central moment is: $\lambda^{1.5}/\sqrt{\lambda} = \lambda$.

$\lambda = E[(X - \lambda)^3] = E[X^3] - 3\lambda E[X^2] + 3\lambda^2 E[X] - \lambda^3$.

$\Rightarrow E[X^3] = \lambda + 3\lambda E[X^2] - 3\lambda^2 E[X] + \lambda^3 = \lambda + 3\lambda(\lambda + \lambda^2) - 3\lambda^2\lambda + \lambda^3 = \lambda^3 + 3\lambda^2 + \lambda$.

$= 125 + 75 + 5 = 205$.

Comment: One could compute enough of the densities and then calculate the third moment:

<table>
<thead>
<tr>
<th>Number of Claims</th>
<th>Probability</th>
<th>Probability x</th>
<th>Probability x</th>
<th>Probability x</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Density Function</td>
<td># of Claims</td>
<td>Square of</td>
<td>Cube of</td>
</tr>
<tr>
<td>0</td>
<td>0.674%</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>1</td>
<td>3.369%</td>
<td>0.03369</td>
<td>0.03369</td>
<td>0.03369</td>
</tr>
<tr>
<td>2</td>
<td>8.422%</td>
<td>0.16845</td>
<td>0.33690</td>
<td>0.67379</td>
</tr>
<tr>
<td>3</td>
<td>14.037%</td>
<td>0.42112</td>
<td>1.26337</td>
<td>3.79010</td>
</tr>
<tr>
<td>4</td>
<td>17.547%</td>
<td>0.70187</td>
<td>2.80748</td>
<td>11.22991</td>
</tr>
<tr>
<td>5</td>
<td>17.547%</td>
<td>0.87734</td>
<td>4.38668</td>
<td>21.93342</td>
</tr>
<tr>
<td>6</td>
<td>14.622%</td>
<td>0.87734</td>
<td>5.26402</td>
<td>31.58413</td>
</tr>
<tr>
<td>7</td>
<td>10.444%</td>
<td>0.73111</td>
<td>5.11780</td>
<td>35.82459</td>
</tr>
<tr>
<td>8</td>
<td>6.528%</td>
<td>0.52222</td>
<td>4.17779</td>
<td>33.42236</td>
</tr>
<tr>
<td>9</td>
<td>3.627%</td>
<td>0.32634</td>
<td>2.93751</td>
<td>26.43761</td>
</tr>
<tr>
<td>10</td>
<td>1.813%</td>
<td>0.18133</td>
<td>1.81328</td>
<td>18.13279</td>
</tr>
<tr>
<td>11</td>
<td>0.824%</td>
<td>0.09066</td>
<td>0.99730</td>
<td>10.97034</td>
</tr>
<tr>
<td>12</td>
<td>0.343%</td>
<td>0.04121</td>
<td>0.49453</td>
<td>5.93437</td>
</tr>
<tr>
<td>13</td>
<td>0.132%</td>
<td>0.01717</td>
<td>0.22323</td>
<td>2.90193</td>
</tr>
<tr>
<td>14</td>
<td>0.047%</td>
<td>0.00660</td>
<td>0.09246</td>
<td>1.29444</td>
</tr>
<tr>
<td>15</td>
<td>0.016%</td>
<td>0.00236</td>
<td>0.03538</td>
<td>0.53070</td>
</tr>
<tr>
<td>16</td>
<td>0.005%</td>
<td>0.00079</td>
<td>0.01258</td>
<td>0.20127</td>
</tr>
<tr>
<td>17</td>
<td>0.001%</td>
<td>0.00025</td>
<td>0.00418</td>
<td>0.07101</td>
</tr>
<tr>
<td>Sum</td>
<td>0.999999458366</td>
<td>4.99990</td>
<td>29.99818</td>
<td>204.96644</td>
</tr>
</tbody>
</table>
10.3. E. \( f(x) = \frac{8!}{(x!(8-x)!)}q^x(1-q)^{8-x} \), for \( x = 0 \) to 8.

\[
E[X(X-1)(X-2)] = \sum_{x=0}^{8} x(x-1)(x-2)f(x) = \sum_{x=3}^{8} x(x-1)(x-2)\left(\frac{8!}{(x!(8-x)!)}q^x(1-q)^{8-x}\right) = (8)(7)(6)q^3 \sum_{x=3}^{5} \left(\frac{5!}{(x-3!(8-x)!)}q^{x-3}(1-q)^{8-x}\right) = 336q^3.
\]

Alternately, the 3rd factorial moment is the 3rd derivative of the p.g.f. at \( z = 1 \).

For the Binomial: \( P(z) = (1 + q(z-1))^m \). \( dP/dz = mq(1 + q(z-1))^{(m-1)} \).

\[
P''(z) = m(m-1)q^2(1 + q(z-1))^{(m-2)} \quad P'''(z) = m(m-1)(m-2)q^3(1 + q(z-1))^{(m-3)}.
\]

Comment: Note that the product \( x(x-1)(x-2) \) is zero for \( x = 0,1 \) and 2, so only terms for \( x \geq 3 \) contribute to the sum. Then a change of variables is made: \( y = x-3 \). Then the resulting sum is the sum of Binomial terms from \( y = 0 \) to 5, which sum is one, since the Binomial is a Distribution, with a support in this case 0 to 5. The expected value of: \( X(X-1)(X-2) \), is an example of what is referred to as a factorial moment.

In the case of the Binomial, the kth factorial moment for \( k \leq m \) is:

\[
p^k(m!)/(m-k)! = p^k(m)(m-1)...(m-(k-1)).
\]

In our case we have the 3rd factorial moment (involving the product of 3 terms) equal to: \( q^3(m)(m-1)(m-2) \).


\( Var[N] = E[N^2] - E[N]^2 = 108 - 10^2 = 8 \).

Comment: Similar to CAS3, 11/06, Q.25.
10.5. C. \( f(x) = \frac{(x+8)!}{(x!)(8!)} \left( \frac{7}{3} \right)^x \left( 1 + \frac{7}{3} \right)^{x+9} \). \( E[ X(X-1)(X-2)(X-3) ] = \sum_{x=0}^{\infty} x(x-1)(x-2)(x-3)f(x) = \sum_{x=0}^{x=4} x(x-1)(x-2)(x-3)f(x) \)

\[
\sum_{x=0}^{\infty} x(x-1)(x-2)(x-3)f(x) = \sum_{x=0}^{x=4} x(x-1)(x-2)(x-3)f(x) = (12)(11)(10)(9)(7/3)^4 \sum_{x=4}^{\infty} ((x+8)! / (x!)(12!))(7/3)^x / (1 + 7/3)^{x+9} = 352,147.
\]

Alternately, the 4th factorial moment is the 4th derivative of the p.g.f. at z = 1.

For the Negative Binomial: \( P(z) = (1 - \beta(z-1))^{-r} \). \( dP/dz = r\beta(z-1)^{-r-1} \). \( P''(z) = r(r+1)\beta^2(z-1)^{-r-2} \). \( P'''(z) = r(r+1)(r+2)\beta^3(z-1)^{-r-3} \). \( P''''(1) = r(r+1)(r+2)(r+3)\beta^4(z-1)^{-r-4} = 352,147. \)

Comments: Note that the product \( x(x-1)(x-2)(x-3) \) is zero for \( x = 0, 1, 2, 3 \), so only terms for \( x \geq 4 \) contribute to the sum. Then a change of variables is made: \( y = x-4 \). Then the resulting sum is the sum of Negative Binomial terms (with \( \beta = 7/3 \) and \( r = 13 \)) from \( y = 0 \) to infinity, which sum is one, since the Negative Binomial is a Distribution with support from 0 to \( \infty \).

The expected value of \( X(X-1)(X-2)(X-3) \), is an example of a factorial moment.

In the case of the Negative Binomial, the mth factorial moment is: \( \beta^m (r)(r+1)...(r+m-1) \).

In our case we have the 4th factorial moment (involving the product of 4 terms) equal to: \( \beta^4 (r)(r+1)(r+2)(r+3) \), with \( \beta = 7/3 \) and \( r = 9 \).
10.6. \( P(z) = \{1 + q(z-1)\}^m = \{1 + 0.3(z-1)\}^{10} = \{0.7 + 0.3z\}^{10} \).

\( P'(z) = (10)(0.3)\{0.7 + 0.3z\}^9 \). \( P''(z) = (3)(2.7)\{0.7 + 0.3z\}^8 \). \( P'''(z) = (3)(2.7)(2.4)\{0.7 + 0.3z\}^7 \).

mean = first factorial moment = \( P'(1) = 3 \).

Second factorial moment = \( P''(1) = (3)(2.7) = 8.1 \).

Third factorial moment = \( P'''(1) = (3)(2.7)(2.4) = 19.44 \).

\( P(z) = \{1 - \beta(z-1)\}^{-r} = \{1 - 3(z-1)\}^{-10} = (4 - 3z)^{-10} \).

\( P'(z) = (-10)(-3)(4 - 3z)^{-11} \). \( P''(z) = (30)(33)(4 - 3z)^{-12} \). \( P'''(z) = (30)(33)(36)(4 - 3z)^{-13} \).

mean = first factorial moment = \( P'(1) = 30 \).

Second factorial moment = \( P''(1) = (30)(33) = 990 \).

Third factorial moment = \( P'''(1) = (30)(33)(36) = 35,640 \).

\( \Rightarrow E[X^2] = 8.1 + 3 = 11.1 \).

\( E[X^3] = 19.44 + (3)(11.1) - (2)(3) = 46.74 \).

Comment: \( E[X^2] = \text{variance} + \text{mean}^2 = 2.1 + 3^2 = 11.1 \).

One could compute all of the densities and then calculate the third moment:

<table>
<thead>
<tr>
<th>Number of Claims</th>
<th>Density Function</th>
<th>Probability</th>
<th>Probability x</th>
<th>Probability x</th>
<th>Square of Claims</th>
<th>Cube of Claims</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.825%</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>12.106%</td>
<td>0.12106</td>
<td>0.12106</td>
<td>0.12106</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>23.347%</td>
<td>0.46695</td>
<td>0.93390</td>
<td>1.86780</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>26.683%</td>
<td>0.80048</td>
<td>2.40145</td>
<td>7.20435</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>20.012%</td>
<td>0.80048</td>
<td>3.20194</td>
<td>12.80774</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10.292%</td>
<td>0.51460</td>
<td>2.57298</td>
<td>12.86492</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3.676%</td>
<td>0.22054</td>
<td>0.44108</td>
<td>3.08758</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.900%</td>
<td>0.06301</td>
<td>0.09259</td>
<td>0.09259</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.145%</td>
<td>0.01157</td>
<td>0.01116</td>
<td>0.10044</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.014%</td>
<td>0.00124</td>
<td>0.00059</td>
<td>0.00059</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.001%</td>
<td>0.00006</td>
<td>0.00006</td>
<td>0.00006</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sum</td>
<td>1</td>
<td>3.00000</td>
<td>11.10000</td>
<td>46.74000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

10.7. \( P(z) = \{1 - \beta(z-1)\}^{-r} = \{1 - 3(z-1)\}^{-10} = (4 - 3z)^{-10} \).

\( P'(z) = (-10)(-3)(4 - 3z)^{-11} \). \( P''(z) = (30)(33)(4 - 3z)^{-12} \). \( P'''(z) = (30)(33)(36)(4 - 3z)^{-13} \).

mean = first factorial moment = \( P'(1) = 30 \).

Second factorial moment = \( P''(1) = (30)(33) = 990 \).

Third factorial moment = \( P'''(1) = (30)(33)(36) = 35,640 \).

\( \Rightarrow E[X^2] = 990 + 30 = 1020 \).

\( E[X^3] = 35,640 + (3)(1020) - (2)(30) = 38,640 \).

Comment: \( E[X^2] = \text{variance} + \text{mean}^2 = (10)(3)(4) + 30^2 = 1020 \).
10.8. D. For a discrete distribution, the expected value of a quantity is determined by taking the sum of its product with the probability density function. In this case, the density of the Poisson is: 
\[ e^{-\lambda} \frac{\lambda^x}{x!}, \ x = 0, 1, 2... \]
Thus \[ E[X(X -1)...(X - 9)] = \]
\[ \infty \infty \infty \sum (e^{-\lambda} \frac{\lambda^x}{x!}) \frac{x(x -1)...(x - 9)}{x!} = e^{-\lambda} \frac{\lambda^10}{10!} \sum \frac{\lambda y}{y!} = e^{-\lambda} \frac{\lambda^10}{10!} \lambda^{10} = \lambda^{10}. \]
Alternately, the 10th factorial moment is the 10th derivative of the p.g.f. at \( z = 1 \).
For the Poisson: \( P(z) = \exp(\lambda(z-1)). \) \( dP/dz = \lambda \exp(\lambda(z-1)). \) \( P''(z) = \lambda^2 \exp(\lambda(z-1)). \)
\( P'''(z) = \lambda^3 \exp(\lambda(z-1)). \) \( P^{10}(z) = \lambda^{10} \exp(\lambda(z-1)). \) \( P^{10}(1) = \lambda^{10}. \)
Comment: Note that the product \( x(x -1)...(x - 9) \) is zero for \( x = 0,1...,9 \), so only terms for \( x \geq 10 \) contribute to the sum. The expected value of \( X(X -1)...(X - 9) \), is an example of a factorial moment. In the case of the Poisson, the nth factorial moment is \( \lambda \) to the nth power. In our case we have the 10th factorial moment (involving the product of 10 terms) equal to \( \lambda^{10}. \)

\( \text{Var}[N] = E[N^2] - E[N]^2 = 8 - 2^2 = 4. \)
Comment: \( P(z) = E[z^n] = \sum f(n)z^n. \) \( P'(1) = \sum nf(n) = E[N]. \)
\( P''(1) = \sum n(n-1)f(n) = E[N(N-1)]. \)

11.1. E. 1. The variance of the Negative Binomial Distribution is greater than the mean. Thus Statement #1 is false. 2. The variance of the Poisson always exists (and is equal to the mean.) Thus Statement #2 is false. 3. The variance of the Binomial Distribution is less than the mean. Thus Statement #3 is false.

11.2. A. For \( a < 0 \), one has a Binomial Distribution.
Comment: Since \( a = -q/(1-q), \) \( q = a/(a-1) = -2/(-3) = 2/3. \)
\( a = 0 \) is a Poisson, \( 1 > a > 0 \) is a Negative Binomial. The Logarithmic Distribution is not a member of the \((a,b,0)\) class. The Logarithmic Distribution is a member of the \((a,b,1)\) class.

11.3. B. \( f(x+1) = f(x) \{a + b/(x+1)\} = f(x)\{.4 + 2/(x+1)\} = f(x)(.4)(x+6)/(x+1). \)
Then proceed iteratively. For example \( f(5) = f(4)(.4)(10)/5 = (.1505)(.8) = .1204. \)

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(n)</td>
<td>0.0467</td>
<td>0.1120</td>
<td>0.1568</td>
<td>0.1672</td>
<td>0.1505</td>
<td>0.1204</td>
<td>0.0883</td>
<td>0.0605</td>
</tr>
</tbody>
</table>

Comment: Since \( 0 < a < 1 \) we have a Negative Binomial Distribution. \( r = 1 + b/a = 1 + (2/.4) = 6. \)
\( \beta = a/(1-a) = .4/.6 = 2/3. \) Thus once \( a \) and \( b \) are given in fact \( f(4) \) is determined. Normally one would compute \( f(0) = (1+\beta)^{-1} = .6^6 = .0467, \) and proceed iteratively from there.
11.4. E. For a member of the (a, b, 0) class of distributions, \( f(x+1) / f(x) = a + \{b / (x+1)\} \).

\[
f(2)/f(1) = a + b/2. \quad \Rightarrow \quad .0512/.0064 = 8 = a + b/2.
\]

\[
f(3)/f(2) = a + b/3. \quad \Rightarrow \quad .2048/.0512 = 4 = a + b/3.
\]

Therefore, \(a = -4\) and \(b = 24\).

\[
f(4) = f(3)(a + b/4) = (.2048)(-4 + 24/4) = .4096.
\]

Comment: Similar to 3, 11/02, Q.28.

11.5. B. This is a member of the (a, b, 0) class of frequency distributions with \(a = 1/3\) and \(b = 0.6\). Since \(a > 0\), this is a Negative Binomial, with \(a = \beta/(1+\beta) = 1/3\), and \(b = (r - 1)\beta/(1 + \beta) = .6\). Therefore, \(r - 1 = .6/(1/3) = 1.8 \Rightarrow r = 2.8\). \(\beta = 0.5\).

\[
f(3) = \{(2.8)(3.8)(4.8)/3!\} .5^3/(1.5^{2.8+3}) = .1013.
\]

Comment: Similar to 3, 5/01, Q.25. \(f(x+1) = f(x) \{a + b/(x+1)\}\), \(x = 0, 1, 2, ...\)

11.6. D. For a member of the (a, b, 0) class, the mode is the largest integer in \(b/(1-a) = 2.8/(1-.4) = 4.667\). Therefore, the mode is 4.

Alternately, \(f(x+1)/f(x) = a + b/(x+1) = .4 + 2.8/(x+1)\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x+1)/f(x))</td>
<td>3.200</td>
<td>1.800</td>
<td>1.333</td>
<td>1.100</td>
<td>0.960</td>
<td>0.867</td>
<td>0.800</td>
</tr>
</tbody>
</table>

Therefore, \(f(4) = 1.1 f(3) > f(3)\), but \(f(5) = .96f(4) < f(4)\). Therefore, the mode is 4.

Alternately, since \(a > 0\), this a Negative Binomial Distribution with \(a = \beta/(1+\beta)\) and \(b = (r-1)\beta/(1+\beta)\). Therefore, \(\beta = a/(1-a) = .4/.6 = 2/3\) and \(r = b/a + 1 = 2.8/.4 + 1 = 8\).

The mode of a Negative Binomial is the largest integer in: \((r-1)\beta = (7)(2/3) = 4.6667\). Therefore, the mode is 4.

11.7. E. This is a member of the (a, b, 0) class of frequency distributions with \(a = -2/3\) and \(b = 4\). Since \(a < 0\), this is a Binomial, with \(a = -q/(1-q) = -2/3\), and \(b = (m+1)q/(1-q) = 4\).

Therefore, \(m + 1 = 4/(2/3) = 6; m = 5\). \(q = .4\). \(f(3) = \{(5!)/(3!(2!))\} .4^3 .6^2 = .2304\).

Comment: Similar to 3, 5/01, Q.25. \(f(x) = f(x-1) \{a + b/x\}, x = 1, 2, 3, ...\)

11.8. B. For a member of the (a,b,0) class of distributions, \(f(x+1) / f(x) = a + \{b / (x+1)\}\).

\[
f(101)/f(100) = a + b/101. \quad \Rightarrow \quad .0329445/0.0350252 = .940594 = a + b/101.
\]

\[
f(102)/f(101) = a + b/102. \quad \Rightarrow \quad .0306836 /0.0329445 = .931372 = a + b/102.
\]

Therefore, \(a = 0\) and \(b = 95.0\). \(f(105) = f(102)(a + b/103)(a + b/104)(a + b/105) = (0.0306836)(95/103)(95/104)(95/105) = .0233893\).

Comment: Alternately, once one solves for a and b, \(a = 0 \Rightarrow a\ Poisson\ Distribution\).

\(\lambda = b = 95\). \(f(105) = e^{-95}(95^{105})(105!) = .0233893\), difficult to calculate using most calculators.
11.9. D. The Binomial is the only member of the \((a, b, 0)\) class with finite support.

\[ P(X = 11) = 0 \quad \text{and} \quad P(X = 10) > 0 \implies m = 10. \]

\[ .1074 = P(X = 10) = q^{10} \quad \Rightarrow q = .800. \]

\[ P(X = 6) = 10!/6!4! (1-q)^4 q^6 = (210) .2^4 .8^6 = \textbf{.088}. \]

11.10. \( f(1) = f(0) (a + b). \) \( f(2) = f(1) (a + b/2). \) \( f(3) = f(2) (a + b/3). \) \( f(4) = f(3) (a + b/4), \) etc.

For \( a = -2 \) and \( b = 6: \)

\[ f(1) = f(0) (-2 + 6) = 4 f(0). \]

\[ f(2) = f(1) (-2 + 6/2) = f(1). \]

\[ f(3) = f(2) (-2 + 6/3) = 0. \]

\[ f(4) = 0, \text{ etc.} \]

This is a Binomial with \( m = 2 \) and \( q = a/(a-1) = 2/3. \)

\[ f(0) = 1/9. \quad f(1) = 4/9. \quad f(2) = 4/9. \]

For \( a = -2 \) and \( b = 5: \)

\[ f(1) = f(0) (-2 + 5) = 3 f(0). \]

\[ f(2) = f(1) (-2 + 5/2) = 1.5f(1). \]

\[ f(3) = f(2) (-2 + 5/3) < 0. \]

No good!

\textit{Comment: Similar to Exercise 6.3 in Loss Models.}

For \( a < 0, \) we require that \( b/a \) be a negative integer.

11.11. B. This is the \((a, b, 0)\) relationship, with \( a = -c \) and \( b = 4c. \)

For the Binomial, \( a < 0. \) For the Poisson \( a = 0. \) For the Negative Binomial, \( a > 0. \)

\( c \) must be positive, since the densities are positive, therefore, \( a < 0 \) and this is a Binomial. For the Binomial, \( a = -q/(1-q) \) and \( b = (m+1)q/(1-q). \)

\[ b = -4a. \quad \Rightarrow m + 1 = 4. \quad \Rightarrow m = 3. \]

\[ 0.7 = p_0 = (1 - q)^m = (1 - q)^3. \quad \Rightarrow q = .1121. \]

\[ c = -a = q/(1-q) = .1121/.8879 = \textbf{.126}. \]

\textit{Comment: Similar to SOA M, 5/05, Q.19.}

11.12. \( f(1) = f(0) (1 - 1/2) = (1/2) f(0). \)

\( f(2) = f(1) (1 - 1/4) = (3/4)f(1). \)

\[ f(3) = f(2) (1 - 1/6) = (5/6)f(2). \]

\[ f(4) = f(3) (1 - 1/8) = (7/8)f(3). \]

\[ f(5) = f(4) (1 - 1/10) = (9/10)f(4). \]

The sum of these densities is:


\[ f(0)(1 + 1/2 + 3/8 + 5/16 + 35/128 + 315/1280 + \ldots) > f(0)(1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + \ldots). \]

However, the sum \( 1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + \ldots, \) diverges.

Therefore, these densities would sum to infinity.

\textit{Comment: We require that \( a < 1. \ a \) is positive for a Negative Binomial; \( a = \beta/(1 + \beta) < 1. \)
11.13. For a member of the \((a, b, 0)\) class \(f(1)/f(0) = a + b\), and \(f(2)/f(1) = a + b/2\).
\[
a + b = 2586.27/6590.79 = 0.39241.
\]
\[
a + b/2 = 656.41/2586.27 = 0.25381.
\]
\[\Rightarrow b = 0.27720. \Rightarrow a = 0.11521.\]

Looking in Appendix B in the tables attached to the exam, \(a\) is positive for the Negative Binomial. Therefore, we have a Negative Binomial.
\[
0.11521 = a = \beta/(1+\beta).
\]
\[\Rightarrow 1/\beta = 1/0.11521 - 1 = 7.6798. \Rightarrow \beta = 0.1302.\]
\[
0.27720 = b = (r-1) \beta/(1+\beta).
\]
\[\Rightarrow r - 1 = 0.27720/0.11521 = 2.4060. \Rightarrow r = 3.406.\]

Comment: Similar to Exercise 16.21b in Loss Models.

2. False. The variance = \(nq(1-q)\) is less than the mean = \(nq\), since \(q < 1\). 
3. True. Statement 3 is referring to the mixture of Poissons via a Gamma, which results in a Negative Binomial frequency distribution for the entire portfolio.

11.15. B. The mean frequency is .5 and the variance is: \(0.75 - .5^2 = .5\).

<table>
<thead>
<tr>
<th>Number of Insureds</th>
<th>Number of Claims</th>
<th>Square of Number of Claims</th>
</tr>
</thead>
<tbody>
<tr>
<td>6070</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3022</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>764</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>126</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>18</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>Average</td>
<td>0.5000</td>
<td>0.7500</td>
</tr>
</tbody>
</table>

Since estimated mean = estimated variance, we expect the Poisson to provide the best fit. 

Comment: If the estimated mean is approximately equal to the estimated variance, then the Poisson is likely to provide a good fit. The Pareto and the LogNormal are continuous distributions not used to fit discrete frequency distributions.

11.16. A. \(\text{Var}[X] = E[X^2] - E[X]^2 = 8.16 - 2.40^2 = 2.4 = E[X]\), so a Poisson Distribution is a good choice for \(X\). \(\text{Var}[Y] = E[Y^2] - E[Y]^2 = 20.25 - 3.50^2 = 8 > 3.5 = E[Y]\), so a Negative Binomial Distribution is a good choice for \(Y\).
11.17. Mean frequency = $500,000/$5000 = 100. Assuming frequency and severity are independent: Var[S] = 7.5 x 10^9 = (100)(5000^2) + (5000^2) (Variance of the frequency).

Variance of the frequency = 200. Thus if each insured has the same frequency distribution, then it has variance > mean, so it might be a Negative Binomial. Alternately, each insured could have a Poisson frequency, but with the means varying across the portfolio. In that case, the mean of mixing distribution = 100. When mixing Poissons, Variance of the mixed distribution

= Mean of mixing Distribution + Variance of the mixing distribution,
so the variance of the mixing distribution = 200 - 100 = 100.

Comment: There are many possible other answers.

11.18. C. \( f(x+1)/f(x) = 2/(x+1) \), \( x = 0, 1, 2, \ldots \)
This is a member of the \((a, b, 0)\) class of frequency distributions:
with \( f(x+1)/f(x) = a + b/(x+1) \), for \( a = 0 \) and \( b = 2 \).

Since \( a = 0 \), this is a Poisson with \( \lambda = b = 2 \). \( f(4) = e^{-2} \frac{2^4}{4!} = 0.090 \).

Alternately, let \( f(0) = c \). Then \( f(1) = 2c \), \( f(2) = 2^2 c/2! \), \( f(3) = 2^3 c/3! \), \( f(4) = 2^4 c/4! \), ...\n
\[ 1 = \sum f(i) = \sum \frac{2^i c}{i!} = c \sum \frac{2^i}{i!} = ce^2. \]
Therefore, \( c = e^{-2} \). \( f(4) = e^{-2} \frac{2^4}{4!} = 0.090 \).

11.19. B. For a member of the \((a, b, 0)\) class of distributions, \( f(x+1) / f(x) = a + \{b / (x+1)\} \).
\( f(1)/f(0) = a + b. \quad \Rightarrow \quad .25/.25 = 1 = a + b. \)
\( f(2)/f(1) = a + b/2. \quad \Rightarrow \quad .1875/.25 = .75 = a + b/2. \)
Therefore, \( a = .5 \) and \( b = .5 \).
\( f(3) = f(2)(a + b/3) = (.1875)(.5 + .5/3) = .125. \)

Alternately, once one solves for \( a \) and \( b \), \( a > 0 \Rightarrow \) a Negative Binomial Distribution.
\( 1/2 = a = \beta/(1 + \beta). \quad \Rightarrow \quad \beta = 1. \quad 1/2 = b = (r-1)\beta/(1 + \beta). \quad \Rightarrow \quad r - 1 = 1. \quad \Rightarrow \quad r = 2. \)
\( f(3) = r(r + 1)(r + 2) \beta^3/\{(1 + \beta)^r+3 3!\} = (2)(3)(4)/\{(2^5)(6)\} = .125. \)

11.20. C. 1. True. 2. False. Would be true if \( \beta = \beta' \), in which case the sum would have the sum of the \( r \) parameters. 3. True. The sum would have the sum of the \( m \) parameters.

Comment: Note the requirement that the variables be independent.

11.21. A. The sum of two independent negative binomial distributions with parameters \( (r_1, \beta_1) \) and \( (r_2, \beta_2) \) is negative binomial if and only if \( \beta_1 = \beta_2 \). Statement 1 is false.
The sum of two independent binomial distributions with parameters \( (q_1, m_1) \) and \( (q_2, m_2) \) is binomial if and only if \( q_1 = q_2 \). Statement 2 is false.
The sum of two independent Poison distributions with parameters \( \lambda_1 \) and \( \lambda_2 \) is Poison, \textit{regardless} of the values of lambda. Statement 3 is false.
11.22. C. This is the (a, b, 0) relationship, with \( a = c \) and \( b = c \).

For the Binomial, \( a < 0 \). For the Poisson \( a = 0 \). For the Negative Binomial, \( a > 0 \).

\( c \) must be positive, since the densities are positive, therefore, \( a > 0 \) and this is a Negative Binomial. For the Negative Binomial, \( a = \frac{\beta}{1+\beta} \) and \( b = \frac{(r-1)\beta}{1+\beta} \).

\[ a = b \Rightarrow r - 1 = 1. \Rightarrow r = 2. \]

\[ 0.5 = p_0 = 1/(1+\beta)^r = 1/(1+\beta)^2. \Rightarrow (1+\beta)^2 = 2. \Rightarrow \beta = \sqrt{2} - 1 = .4142. \]

\[ c = a = \frac{\beta}{1+\beta} = .4142/1.4142 = 0.293. \]

11.23. C. For a member of the (a, b, 0) class, \( f(1)/f(0) = a + b \), and \( f(2)/f(1) = a + b/2 \).

Therefore, \( a + b = 1 \), and \( a + b/2 = 0.196608/0.327680 = .6 \). \( \Rightarrow a = 0.2 \) and \( b = 0.8 \).

Since \( a \) is positive, we have a Negative Binomial Distribution. Statement III is true.

\[ f(3) = f(2)(a + b/3) = (0.196608)(0.2 + 0.8/3) = 0.0917504. \] Statement I is false.

Comment: \( .2 = a = \frac{\beta}{1+\beta} \) and \( .8 = b = \frac{(r-1)\beta}{1+\beta} \). \( \Rightarrow r = 5 \) and \( \beta = 0.25 \).

\[ E[N] = r\beta = (5)(.25) = 1.25, \text{ as given.} \]

\[ f(3) = \frac{r(r+1)(r+2)/3!}{1+\beta}^3/(1+\beta)^{3+r} = \frac{(5)(6)(7)/6}{1.25^8} = 0.0917504. \]

12.1. C. Calculate \((x+1)f(x+1)/f(x)\). Since it is approximately linear, we seem to have a member of the (a, b, 0) class. \( f(x+1)/f(x) = a + b/(x+1) \), so \((x+1)f(x+1)/f(x) = a(x+1) + b = ax + a + b. The slope is positive, so \( a > 0 \) and we have a **Negative Binomial**. The slope, \( a \approx .17 \). The intercept is about .08. Thus \( a + b \approx .08 \).

Therefore, \( b \approx .08 - .17 = -.09 < 0 \).

For the Negative Binomial \( b = \frac{(r-1)\beta}{1+\beta} \). Thus \( b < 0 \), implies \( r < 1 \).

<table>
<thead>
<tr>
<th>Number of Accident</th>
<th>Observed Density</th>
<th>((x+1)f(x+1)/f(x)) Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>91,304</td>
<td>0.91304</td>
</tr>
<tr>
<td>1</td>
<td>7,586</td>
<td>0.07586</td>
</tr>
<tr>
<td>2</td>
<td>955</td>
<td>0.00955</td>
</tr>
<tr>
<td>3</td>
<td>133</td>
<td>0.00133</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>0.00018</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>0.00003</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0.00001</td>
</tr>
<tr>
<td>7+</td>
<td>0</td>
<td>0.00000</td>
</tr>
</tbody>
</table>

Comment: Similar to 4, 5/00, Q.40. Do not put much weight on the values of \((x+1)f(x+1)/f(x)\) in the righthand tail, that can be greatly affected by random fluctuation.

The first moment is .09988, and the second moment is .13002.

The variance is: .13002 - .09988^2 = .12004, significantly greater than the mean.
12.2. B. Calculate \((x+1)f(x+1)/f(x)\). Since it is approximately linear, we seem to have a member of the \((a, b, 0)\) class. \(f(x+1)/f(x) = a + b/(x+1)\), so \((x+1)f(x+1)/f(x) = a(x+1) + b = ax + a + b\). The slope seems close to zero, until the data starts to get thin, so \(a \approx 0\) and therefore we assume this data probably came from a \textbf{Poisson}.

<table>
<thead>
<tr>
<th>Accident</th>
<th>Observed</th>
<th>Density</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.0860</td>
</tr>
<tr>
<td>1</td>
<td>2,057</td>
<td>0.2057</td>
</tr>
<tr>
<td>2</td>
<td>2,506</td>
<td>0.2506</td>
</tr>
<tr>
<td>3</td>
<td>2,231</td>
<td>0.2231</td>
</tr>
<tr>
<td>4</td>
<td>1,279</td>
<td>0.1279</td>
</tr>
<tr>
<td>5</td>
<td>643</td>
<td>0.0643</td>
</tr>
<tr>
<td>6</td>
<td>276</td>
<td>0.0276</td>
</tr>
<tr>
<td>7</td>
<td>101</td>
<td>0.0101</td>
</tr>
<tr>
<td>8</td>
<td>41</td>
<td>0.0041</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>0.0004</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

Comment: Any actual data set is subject to random fluctuation, and therefore the observed slope of the accident profile will never be exactly zero. One can never distinguish between the possibility that the model was a Binomial with \(q\) small, a Poisson, or a Negative Binomial with \(\beta\) small. This data was simulated as 10,000 independent random draws from a Poisson with \(\lambda = 2.5\).

12.3. E. Calculate \((x+1)f(x+1)/f(x)\).

Note that \(f(x+1)/f(x) = (\text{number with } x + 1)/(\text{number with } x)\).

Since \((x+1)f(x+1)/f(x)\) is not linear, we do not have a member of the \((a, b, 0)\) class.

<table>
<thead>
<tr>
<th>Number of runs</th>
<th>Observed</th>
<th>((x+1)f(x+1)/f(x))</th>
<th>Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>518,228</td>
<td>0.203</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>105,070</td>
<td>0.912</td>
<td>0.710</td>
</tr>
<tr>
<td>2</td>
<td>47,936</td>
<td>1.356</td>
<td>0.444</td>
</tr>
<tr>
<td>3</td>
<td>21,673</td>
<td>1.797</td>
<td>0.441</td>
</tr>
<tr>
<td>4</td>
<td>9,736</td>
<td>2.071</td>
<td>0.274</td>
</tr>
<tr>
<td>5</td>
<td>4,033</td>
<td>2.513</td>
<td>0.442</td>
</tr>
<tr>
<td>6</td>
<td>1,689</td>
<td>2.648</td>
<td>0.136</td>
</tr>
<tr>
<td>7</td>
<td>639</td>
<td>3.430</td>
<td>0.782</td>
</tr>
<tr>
<td>8</td>
<td>274</td>
<td>3.515</td>
<td>0.084</td>
</tr>
<tr>
<td>9</td>
<td>107</td>
<td>3.364</td>
<td>-0.150</td>
</tr>
<tr>
<td>10</td>
<td>36</td>
<td>7.639</td>
<td>4.274</td>
</tr>
<tr>
<td>11</td>
<td>25</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Comment: At high numbers of runs, where the data starts to thin out, one would not put much reliance on the values of \((x+1)f(x+1)/f(x)\). The data is taken from “An Analytic Model for Per-inning Scoring Distributions,” by Keith Woolner.
12.4. E. Calculate \((x+1)f(x+1)/f(x)\). Since it does not appear to be linear, we do not seem to have a member of the \((a, b, 0)\) class.

<table>
<thead>
<tr>
<th>Accident</th>
<th>Observed</th>
<th>Density</th>
<th>((x+1)f(x+1)/f(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>820</td>
<td>0.0820</td>
<td>1.677</td>
</tr>
<tr>
<td>1</td>
<td>1,375</td>
<td>0.1375</td>
<td>3.245</td>
</tr>
<tr>
<td>2</td>
<td>2,231</td>
<td>0.2232</td>
<td>2.580</td>
</tr>
<tr>
<td>3</td>
<td>1,919</td>
<td>0.1920</td>
<td>2.912</td>
</tr>
<tr>
<td>4</td>
<td>1,397</td>
<td>0.1397</td>
<td>3.586</td>
</tr>
<tr>
<td>5</td>
<td>1,002</td>
<td>0.1002</td>
<td>4.078</td>
</tr>
<tr>
<td>6</td>
<td>681</td>
<td>0.0681</td>
<td>3.392</td>
</tr>
<tr>
<td>7</td>
<td>330</td>
<td>0.0330</td>
<td>4.170</td>
</tr>
<tr>
<td>8</td>
<td>172</td>
<td>0.0172</td>
<td>2.930</td>
</tr>
<tr>
<td>9</td>
<td>56</td>
<td>0.0056</td>
<td>2.500</td>
</tr>
<tr>
<td>10</td>
<td>14</td>
<td>0.0014</td>
<td>2.357</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
<td>0.0003</td>
<td></td>
</tr>
</tbody>
</table>

12.5. A. Calculate \((x+1)f(x+1)/f(x) = (x+1)(\text{number with } x + 1)/(\text{number with } x)\).

<table>
<thead>
<tr>
<th>Number of claims</th>
<th>Observed</th>
<th>((x+1)f(x+1)/f(x))</th>
<th>Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6,503</td>
<td>1.261</td>
<td>-0.262</td>
</tr>
<tr>
<td>1</td>
<td>8,199</td>
<td>0.999</td>
<td>-0.212</td>
</tr>
<tr>
<td>2</td>
<td>4,094</td>
<td>0.786</td>
<td>-0.309</td>
</tr>
<tr>
<td>3</td>
<td>1,073</td>
<td>0.477</td>
<td>-0.360</td>
</tr>
<tr>
<td>4</td>
<td>128</td>
<td>0.117</td>
<td>-0.360</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Since \((x+1)f(x+1)/f(x)\) is approximately linear, we have a member of the \((a, b, 0)\) class.

\(a = \text{slope} < 0. \Rightarrow \text{Binomial Distribution.}\)

Comment: The data was simulated from a Binomial Distribution with \(m = 5\) and \(q = 0.2\).

12.6. E. Calculate \((x+1)f(x+1)/f(x)\).

Note that \(f(x+1)/f(x) = (\text{number with } x + 1)/(\text{number with } x)\).

<table>
<thead>
<tr>
<th>Number of claims</th>
<th>Observed</th>
<th>((x+1)f(x+1)/f(x))</th>
<th>Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>565,664</td>
<td>0.121</td>
<td>0.029</td>
</tr>
<tr>
<td>1</td>
<td>68,714</td>
<td>0.151</td>
<td>0.061</td>
</tr>
<tr>
<td>2</td>
<td>5,177</td>
<td>0.212</td>
<td>0.052</td>
</tr>
<tr>
<td>3</td>
<td>365</td>
<td>0.263</td>
<td>0.987</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>1.250</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Even ignoring the final value, \((x+1)f(x+1)/f(x)\) is not linear.

Therefore, we do not have a member of the \((a, b, 0)\) class.

Comment: Data taken from Table 6.6.2 in Introductory Statistics with Applications in General Insurance by Hossack, Pollard and Zehnwirth. See also Table 6.5 in Loss Models.
12.7. A. Calculate \((x+1)f(x+1)/f(x)\). Since it seems to be decreasing linearly, we seem to have a member of the \((a, b, 0)\) class, with \(a < 0\), which is a Binomial Distribution.

<table>
<thead>
<tr>
<th>Accident</th>
<th>Observed</th>
<th>Density Function</th>
<th>((x+1)f(x+1)/f(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
<td>0.10000</td>
<td>2.67</td>
</tr>
<tr>
<td>1</td>
<td>267</td>
<td>0.26700</td>
<td>2.33</td>
</tr>
<tr>
<td>2</td>
<td>311</td>
<td>0.31100</td>
<td>2.01</td>
</tr>
<tr>
<td>3</td>
<td>208</td>
<td>0.20800</td>
<td>1.67</td>
</tr>
<tr>
<td>4</td>
<td>87</td>
<td>0.08700</td>
<td>1.32</td>
</tr>
<tr>
<td>5</td>
<td>23</td>
<td>0.02300</td>
<td>1.04</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>0.00400</td>
<td></td>
</tr>
<tr>
<td>7+</td>
<td>0</td>
<td>0.00000</td>
<td></td>
</tr>
</tbody>
</table>

Alternately, the mean is 2, and the second moment is 5.494. Therefore, the sample variance is 
\((1000/999)(5.494 - 2^2) = 1.495. Since the variance is significantly less than the mean, this indicates a Binomial Distribution.

Comment: One would not use a continuous distribution such as the Normal or the Gamma to model a frequency distribution. \((x+1)f(x+1)/f(x) = a(x+1) + b\). In this case, \(a \approx -.33\).

For the Binomial, \(a = -q/ (1-q)\), so \(q \approx .25\). In this case, \(b \approx 2.67+.33 = 3.00\).

For the Binomial, \(b = (m+1)q/ (1-q)\), so \(m \approx (3/.33) -1 = 8\).

12.8. A. Calculate \((x+1)f(x+1)/f(x)\). For example, \((3)(7/84)/(12/84) = (3)(7)/12 = 1.75\).

<table>
<thead>
<tr>
<th>Accident</th>
<th>Observed</th>
<th>((x+1)f(x+1)/f(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>32</td>
<td>0.81</td>
</tr>
<tr>
<td>1</td>
<td>26</td>
<td>0.92</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>1.75</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>2.29</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>2.50</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>3.00</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Since this quantity seems to be increasing roughly linearly, we seem to have a member of the \((a, b, 0)\) class, with \(a = \text{slope} > 0\), which is a Negative Binomial Distribution.

Alternately, the mean is: 103/84 = 1.226, and the second moment is: 287/84 = 3.417.

The sample variance is: \((84/83)(3.417 - 1.226^2) = 1.937. Since the sample variance is significantly more than the sample mean, this indicates a Negative Binomial.

Comment: If \((x+1)f(x+1)/f(x)\) had been approximately linear with a slope that was close to zero, then one could not distinguish between the possibility that the model was a Binomial with \(q\) small, a Poisson, or a Negative Binomial with \(\beta\) small. If the correct model were the discrete uniform, then we would expect the observed number of policies to be similar for each number of accidents.
13.1. B. Let \( f(x) \) be the density of a Poisson Distribution, then the distribution truncated from below at zero is: \( h(x) = \frac{f(x)}{1-f(0)} \). Thus for \( \theta = 0.3 \), \( h(x) = \frac{0.3 \times e^{-0.3}}{x!} / \{1-e^{-0.3}\} \).

\[
h(3) = \frac{0.3 \times e^{-0.3}}{3!} / \{1-e^{-0.3}\} = \frac{0.00333}{0.259} = 1.3%.
\]

13.2. B. Mean is that of the non-truncated binomial, divided by \( 1 - f(0) \):

\[
(0.3)(5) / (1-0.75) = 1.803.
\]

13.3. D. The second moment is that of the non-truncated binomial, divided by \( 1 - f(0) \):

\[
(1.05+1.5^2) / (1-0.75) = 3.967. \text{ Variance } = 3.967 - 1.803^2 = 3.967 - 3.248 = 0.719.
\]

Comment: Using the formula in Appendix B of Loss Models:

\[
\text{Variance } = mq((1-q) - (1 - q + mq)(1-q)^m) / \{1-(1-q)^m\}^2
\]

\[
= (5)(.3)((.7 - (.7 + 1.5)(.7)^5) / (1-(.7)^5)^2 = (1.5)(.3303)/.8319^2 = 0.716.
\]

13.4. D. For a non-truncated binomial, \( f(3) = \frac{5!}{(3!)(2!)} \times 0.3 \times 0.7^2 = 0.1323 \). For the zero-truncated distribution one gets the density by dividing by \( 1 - f(0) \):

\[
\frac{0.1323}{1-0.75} = 15.9%.
\]

13.5. B. For a discrete distribution such as we have here, employ the convention that the median is the first value at which the distribution function is greater than or equal to 0.5.

\[
F(1) = 0.433 < 50\%, F(2) = 0.804 > 50\%, \text{ and therefore the median is } 2.
\]

<table>
<thead>
<tr>
<th>Number of Vehicles</th>
<th>Untruncated Binomial</th>
<th>Zero-Truncated Binomial</th>
<th>Cumulative</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>16.81%</td>
<td>43.29%</td>
<td>Binomial</td>
</tr>
<tr>
<td>1</td>
<td>30.87%</td>
<td>37.11%</td>
<td>80.40%</td>
</tr>
<tr>
<td>2</td>
<td>30.87%</td>
<td>15.90%</td>
<td>96.30%</td>
</tr>
<tr>
<td>3</td>
<td>13.23%</td>
<td>3.41%</td>
<td>99.71%</td>
</tr>
<tr>
<td>4</td>
<td>2.83%</td>
<td>0.29%</td>
<td>100.00%</td>
</tr>
</tbody>
</table>

13.6. E. Mean is that of the non-truncated negative binomial, divided by \(1-f(0)\):

\[
(4)(.5) / (1-1.5^{-4}) = 2 / 0.8025 = 2.49
\]

13.7. D. The second moment is that of the non-truncated negative binomial, divided by \(1-f(0)\):

\[
(3+2^2) / (1-1.5^{-4}) = 8.723. \text{ Variance } = 8.723 - 2.492^2 = 2.51.
\]

Comment: Using the formula in Appendix B of Loss Models:

\[
\text{Variance } = r\beta(1+\beta) - (1 + \beta + r\beta)(1+\beta)^r) / \{1-(1+\beta)^r\}^2
\]

\[
= (4)(.5)((1.5 - (1 + .5 + 2)(1.5^{-4})) / (1-1.5^{-4})^2 = (2)(.8086)/.8025^2 = 2.51.
\]

The non-truncated negative binomial has mean = \(r\beta = 2\), and variance = \(r\beta(1+\beta) = 3\), and thus a second moment of \(3+2^2 = 7\).
13.8. C. For the non-truncated negative binomial, 
\[ f(7) = \frac{(4)(5)(6)(7)(8)(9)(10) \cdot .5^7}{(7!)(1.5^{11})} = 1.08\% \]
For the zero-truncated distribution one gets the density by dividing by 1-f(0): 
\[ \frac{(1.08\%)}{(1-1.5^{-4})} = 1.35\% \]

13.9. D. The chance of more than 5 is: 
\[ 1 - .9471 = 5.29\% \]

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>19.75%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>26.34%</td>
<td>32.82%</td>
<td>32.82%</td>
</tr>
<tr>
<td>2</td>
<td>21.95%</td>
<td>27.35%</td>
<td>60.17%</td>
</tr>
<tr>
<td>3</td>
<td>14.63%</td>
<td>18.23%</td>
<td>78.40%</td>
</tr>
<tr>
<td>4</td>
<td>8.54%</td>
<td>10.64%</td>
<td>89.04%</td>
</tr>
<tr>
<td>5</td>
<td>4.55%</td>
<td>5.67%</td>
<td>94.71%</td>
</tr>
<tr>
<td>6</td>
<td>2.28%</td>
<td>2.84%</td>
<td>97.55%</td>
</tr>
<tr>
<td>7</td>
<td>1.08%</td>
<td>1.35%</td>
<td>98.90%</td>
</tr>
<tr>
<td>8</td>
<td>0.50%</td>
<td>0.62%</td>
<td>99.52%</td>
</tr>
<tr>
<td>9</td>
<td>0.22%</td>
<td>0.28%</td>
<td>99.79%</td>
</tr>
</tbody>
</table>

13.10. A. Mean of the logarithmic distribution is: 
\[ \beta / \ln(1+\beta) = 2 / \ln(3) = 1.82 \]

13.11. B. Variance of the logarithmic distribution is: 
\[ \frac{\beta(1+\beta-\beta/\ln(1+\beta))}{\ln(1+\beta)} = 2(3 - 1.82)/\ln(3) = 2.15. \]

13.12. C. For the logarithmic distribution, 
\[ f(x) = \frac{\beta^{x}}{(1+\beta)^{x}} / \{x \ln(1+\beta)\} \]
\[ f(6) = \frac{(2/3)^6}{\{6 \ln(3)\}} = 1.33\% \]

13.13. A. For the zero-truncated Negative Binomial Distribution, 
\[ f(r) = \frac{\beta^r}{(1+\beta)^r} \cdot \frac{r(r+1)(r+2)(r+3)(r+4)}{(1)(2)(3)(4)(5)} \]
\[ (-.6)(.4)(2.4)(3.4)(3/4)^5 / (120)(4^{-6} - 1) = (-2.742)(.2373) / (120)(-.5647) = .96\%. \]
Comment: Note this is an extended zero-truncated negative binomial distribution, with 
\[ 0 > r > -1. \]
The same formulas apply as when \( r > 0. \) (As \( r \) approaches zero one gets a logarithmic distribution.)
For the untruncated negative binomial distribution we must have \( r > 0. \) So in this case there is no corresponding untruncated distribution.

13.14. D. Mean is that of the non-truncated Poisson, divided by \( 1 - f(0): \)
\[ (2.5) / (1 - e^{-2.5}) = 2.5/.9179 = 2.724. \]
Comment: Note that since the probability at zero has been distributed over the positive integers, 
the mean is larger for the zero-truncated distribution than for the corresponding untruncated distribution.
13.15. A. The second moment is that of the non-truncated Poisson, divided by 1 - f(0):

\[
(2.5 + 2.5^2) / (1 - e^{-2.5}) = 9.533.
\]

Variance = 9.533 - 2.724² = 2.11.

Comment: Using the formula in Appendix B of Loss Models:

\[
\text{Variance} = \lambda \left(1 - (\lambda+1)e^{-\lambda}\right) / (1-e^{-\lambda})^2 = (2.5)(1 - 3.5e^{-2.5})/(1 - e^{-2.5})^2 = (2.5)(.7127)/.9179^2 = 2.11.
\]

13.16. B. For a untruncated Poisson, \( f(6) = (2.5^6)e^{-2.5}/6! = .0278 \). For the zero-truncated distribution one gets the density by dividing by 1-f(0): \((.0278) / (1-e^{-2.5}) = 3.03\% \).

13.17. D. One adds up the chances of 1, 2 and 3 days, and gets 73.59%.

<table>
<thead>
<tr>
<th>Number of Days</th>
<th>Untruncated Poisson</th>
<th>Zero-Truncated Poisson</th>
<th>Cumulative</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.21%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20.52%</td>
<td>22.36%</td>
<td>22.36%</td>
</tr>
<tr>
<td>2</td>
<td>25.65%</td>
<td>27.95%</td>
<td>50.30%</td>
</tr>
<tr>
<td>3</td>
<td>21.38%</td>
<td>23.29%</td>
<td>73.59%</td>
</tr>
<tr>
<td>4</td>
<td>13.36%</td>
<td>14.55%</td>
<td>88.14%</td>
</tr>
<tr>
<td>5</td>
<td>6.68%</td>
<td>7.28%</td>
<td>95.42%</td>
</tr>
<tr>
<td>6</td>
<td>2.78%</td>
<td>3.03%</td>
<td>98.45%</td>
</tr>
<tr>
<td>7</td>
<td>0.99%</td>
<td>1.08%</td>
<td>99.54%</td>
</tr>
<tr>
<td>8</td>
<td>0.31%</td>
<td>0.34%</td>
<td>99.88%</td>
</tr>
</tbody>
</table>

Comment: By definition, there is no probability of zero items for a zero-truncated distribution.

13.18. B. The mode is where the density function is greatest, 2.

<table>
<thead>
<tr>
<th>Number of Days</th>
<th>Untruncated Poisson</th>
<th>Zero-Truncated Poisson</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.21%</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20.52%</td>
<td>22.36%</td>
</tr>
<tr>
<td>2</td>
<td>25.65%</td>
<td>27.95%</td>
</tr>
<tr>
<td>3</td>
<td>21.38%</td>
<td>23.29%</td>
</tr>
<tr>
<td>4</td>
<td>13.36%</td>
<td>14.55%</td>
</tr>
</tbody>
</table>

Comment: Unless the mode of the untruncated distribution is 0, the mode of the zero-truncated distribution is the same as that of the untruncated distribution. For example, in this case all the densities on the positive integers are increased by the same factor \(1/(1 - .0821)\). Thus since the density at 2 was largest prior to truncation, it remains the largest after truncation at zero.

13.19. The number of customers he has to wait is a Zero-Truncated Geometric Distribution with \( \beta = \text{chance of failure} / \text{chance of success} = (1 - .6)/.6 = 1/6 - 1 \).

So the mean number of customers is \(1/6 = 1.67\). \( \Rightarrow 16.7 \text{ minutes} \) on average.

Comment: The mean of the Zero-Truncated Geometric Distribution is:

\[
\beta/(1 - 1/(1+\beta)) = 1 + \beta.
\]
13.20. The number of customers he has to wait is a Zero-Truncated Geometric Distribution with 
\( \beta = \text{chance of failure} / \text{chance of success} = (1 - .4)/.4 = 1/.4 - 1. \)

So the mean number of customers is \( 1/.4 = 2.5. \) \( \Rightarrow \) \textbf{25 minutes} on average.

Comment: Longer patterns can be handled via Markov Chain ideas not on the syllabus.
See Example 4.20 in Probability Models by Ross.

13.21. D. At the end of year one the business has 1700. Thus, if the loss occurs at the end of year one, there is ruin if the size of loss is > 1700, a 70% chance. Similarly, at the end of year 2, if the loss did not occur in year 1, the business has 2700. Thus, if the loss occurs at the end of year two there is ruin if the size of loss is > 2700, a 50% chance.

If the loss occurs at the end of year three there is ruin if the size of loss is > 3700, a 30% chance.
If the loss occurs at the end of year four there is ruin if the size of loss is > 4700, a 10% chance.
If the loss occurs in year 5 or later there is no chance of ruin.

The probability of the loss being in year \( n \) is: 
\[
\left( \frac{1}{1+\beta} \right) \left( \frac{\beta}{1+\beta} \right)^{n-1} = .65(.35^{n-1}).
\]

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Probability of Loss in this year</td>
<td>Probability of Ruin if Loss Occurs in this year</td>
<td>Column B times Column C</td>
</tr>
<tr>
<td>1</td>
<td>0.6500</td>
<td>0.7</td>
<td>0.4550</td>
</tr>
<tr>
<td>2</td>
<td>0.2275</td>
<td>0.5</td>
<td>0.1138</td>
</tr>
<tr>
<td>3</td>
<td>0.0796</td>
<td>0.3</td>
<td>0.0239</td>
</tr>
<tr>
<td>4</td>
<td>0.0279</td>
<td>0.1</td>
<td>0.0028</td>
</tr>
<tr>
<td>5</td>
<td>0.0098</td>
<td>0</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>\textbf{0.5954}</td>
</tr>
</tbody>
</table>

Alternately, if the loss is of size 500, 1000, or 1500 there is not ruin. If the loss is of size 2000 or 2500, then there is ruin if the loss occurs in year 1. If the loss is of size 3000 or 3500, then there is ruin if the loss occurs by year 2. If the loss is of size 4000 or 4500, then there is ruin if the loss occurs by year 3. If the loss is of size 5000, then there is ruin if the loss occurs by year 4.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Size of Loss</td>
<td>Probability of a Loss occurs</td>
<td>Probability that Loss Occurs by</td>
<td>Column B times</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>500, 1000, 1500</td>
<td>0.3</td>
<td>none</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>2000 or 2500</td>
<td>0.2</td>
<td>1</td>
<td>0.650</td>
<td>0.130</td>
</tr>
<tr>
<td>3000 or 3500</td>
<td>0.2</td>
<td>2</td>
<td>0.877</td>
<td>0.175</td>
</tr>
<tr>
<td>4000 or 4500</td>
<td>0.2</td>
<td>3</td>
<td>0.957</td>
<td>0.191</td>
</tr>
<tr>
<td>5000</td>
<td>0.1</td>
<td>4</td>
<td>0.985</td>
<td>0.099</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>\textbf{0.595}</td>
</tr>
</tbody>
</table>

14.1. C. Mean is that of the unmodified Binomial, multiplied by \( (1 - .15) \) and divided by \( 1 - f(0): \)
\[
(.3)(5)(.85) / (1 - .7^5) = \textbf{1.533}.
\]
14.2. D. The second moment is that of the unmodified Binomial, multiplied by (1-.15) and divided by 1-f(0):

\[(1.05+1.5^2)(.85) / (1 - .7^5) = 3.372. \text{ Variance} = 3.372 - 1.533^2 = 1.022.\]

14.3. C. For an unmodified binomial, \(f(3) = (5!/3!(2!)) \cdot .3^3 \cdot .7^2 = .1323.\) For the zero-truncated distribution one gets the density by multiplying by (1-.15) and dividing by 1-f(0):

\[(.1323)(.85) / (1 - .7^5) = 13.5\%\.

14.4. C. The 95th percentile is that value corresponding to the distribution function being 95%. For a discrete distribution such as we have here, employ the convention that the 95th percentile is the first value at which the distribution function is greater than or equal to .95. \(F(2) = .8334 < 95\%,\) \(F(3) = .9686 \geq 95\%,\) and therefore the 95th percentile is 3.

<table>
<thead>
<tr>
<th>Number of Claims</th>
<th>Unmodified Binomial</th>
<th>Zero-Modified Binomial</th>
<th>Cumulative</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>16.81%</td>
<td>15.00%</td>
<td>Binomial</td>
</tr>
<tr>
<td>1</td>
<td>36.02%</td>
<td>36.80%</td>
<td>51.80%</td>
</tr>
<tr>
<td>2</td>
<td>30.87%</td>
<td>31.54%</td>
<td>83.34%</td>
</tr>
<tr>
<td>3</td>
<td>13.23%</td>
<td>13.52%</td>
<td>96.86%</td>
</tr>
<tr>
<td>4</td>
<td>2.83%</td>
<td>2.90%</td>
<td>99.75%</td>
</tr>
<tr>
<td>5</td>
<td>0.24%</td>
<td>0.25%</td>
<td>100.00%</td>
</tr>
</tbody>
</table>

14.5. A. Mean is that of the unmodified negative binomial, multiplied by (1-.35) and divided by 1-f(0): \((4)(.5)(.65)/ (1-1.5^{-4}) = 2 / .8025 = 1.62\)

14.6. E. The second moment is that of the unmodified negative binomial, multiplied by (1-.35) and divided by 1 - f(0): \((3+2^2) (.65)/ (1-1.5^{-4}) = 5.67. \text{ Variance} = 5.67 - 1.62^2 = 3.05.\)

14.7. B. For the unmodified negative binomial, \(f(7) = (4)(5)(6)(7)(8)(9)(10) .5^7 /((7!)(1.5)^{11}) = 1.08\%.\) For the zero-truncated distribution one gets the density by multiplying by (1-.35) and dividing by 1-f(0): \((1.08%)(.65) / (1-1.5^{-4}) = .87\%.\)
14.8. C. The chance of more than 5 claims is: $1 - .9656 = 3.44\%$.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>19.75%</td>
<td>35.00%</td>
<td>Neg. Binomial</td>
</tr>
<tr>
<td>1</td>
<td>26.34%</td>
<td>21.33%</td>
<td>56.33%</td>
</tr>
<tr>
<td>2</td>
<td>21.95%</td>
<td>17.78%</td>
<td>74.11%</td>
</tr>
<tr>
<td>3</td>
<td>14.63%</td>
<td>11.85%</td>
<td>85.96%</td>
</tr>
<tr>
<td>4</td>
<td>8.54%</td>
<td>6.91%</td>
<td>92.88%</td>
</tr>
<tr>
<td>5</td>
<td>4.55%</td>
<td>3.69%</td>
<td>96.56%</td>
</tr>
<tr>
<td>6</td>
<td>2.28%</td>
<td>1.84%</td>
<td>98.41%</td>
</tr>
<tr>
<td>7</td>
<td>1.08%</td>
<td>0.88%</td>
<td>99.29%</td>
</tr>
<tr>
<td>8</td>
<td>0.50%</td>
<td>0.40%</td>
<td>99.69%</td>
</tr>
<tr>
<td>9</td>
<td>0.22%</td>
<td>0.18%</td>
<td>99.87%</td>
</tr>
</tbody>
</table>

14.9. E. Mean of the logarithmic distribution is: $\beta/\ln(1+\beta) = 2/\ln(3) = 1.82$.  
For the zero-modified distribution, the mean is multiplied by $1-.25$: $(.75)(1.82) = 1.37$.  
Comment: Note the unmodified logarithmic distribution has no chance of zero claims.  
Therefore, we need not divide by $1-f(0)$ to get to the zero-modified distribution (or alternately we are dividing by $1 - 0 = 1$.)

14.10. C. Variance of the unmodified logarithmic distribution is: 
$$\beta\{1 + \beta - \beta/\ln(1+\beta)\}/\ln(1+\beta) = 2\{3 -1.82\}/\ln(3) = 2.15.$$  
Thus the unmodified logarithmic has a second moment of: $2.15 + 1.82^2 = 5.46$.  
For the zero-modified distribution, the second moment is multiplied by $1-.25$: $(.75)(5.46) = 4.10$.  
Thus the variance of the zero-modified distribution is: $4.10 - 1.37^2 = 2.22$.

14.11. A. For the unmodified logarithmic distribution, 
$$f(x) = \{\beta/ (1+\beta)\}^x / \{x \ln(1+\beta)\}$$  
f(6) = $(2/3)^6 / \{6\ln(3)\} = 1.33\%$.  
For the zero-modified distribution, the density at 6 is multiplied by $1-.25$: $(.75)(1.33\%) = 1.00\%$.

14.12. A. For the zero-truncated Negative Binomial Distribution, 
$$f(5) = r(r+1)(r+2)(r+3)(r+4) \left(\beta/(1+\beta)\right)^5 / \{(5!)((1+\beta)^r -1)\} = \right.$$
$$(-.6)(.4)(1.4)(2.4)(3.4)(3/4)^5 / \{(120)(4-.6 -1) = (-2.742)(.2373) / (120)(-.5647) = .96\%.$$  
For the zero-modified distribution, multiply by $1 -.2$: $(.8)(.96\%) = .77\%$.  
Comment: Note this is an extended zero-truncated negative binomial distribution, with $0 > r > -1$. The same formulas apply as when $r > 0$. (As $r$ approaches zero one gets a logarithmic distribution.) For the unmodified negative binomial distribution we must have $r > 0$. So in this case there is no corresponding unmodified distribution.
14.13. A. The mean is that of the non-modified Poisson, multiplied by (1-.3) and divided by 1- f(0): (2.5) (.7) / (1-e^{-2.5}) = 1.907.

14.14. E. The second moment is that of the unmodified Poisson, multiplied by (1-.3) and divided by 1-f(0): (2.5+2.5^2)(.7) / (1-e^{-2.5}) = 6.673. Variance = 6.673 - 1.907^2 = 3.04.

14.15. B. For an unmodified Poisson, f(6) = (2.5^6)e^{-2.5}/6! = .0278. For the zero-modified distribution one gets the density by multiplying by (1 - .3) and dividing by 1 - f(0): (.0278)(.7) / (1 - e^{-2.5}) = 2.12%.

14.16. B. For the unmodified Poisson f(0) = e^{-2.5} = 8.208%, and f(2) = 2.5^2e^{-2.5}/2 = 25.652%. The zero-modified Poisson has a density at 2 of: (25.652%)(1 - 30%)/(1 - 8.208%) = 19.56%.

14.17. D. One adds up the chances of 0, 1, 2 and 3 claims, and gets 81.5%.

<table>
<thead>
<tr>
<th>Number of Claims</th>
<th>Unmodified Poisson</th>
<th>Zero-Modified Poisson</th>
<th>Cumulative</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.21%</td>
<td>30.00%</td>
<td>Poisson</td>
</tr>
<tr>
<td>1</td>
<td>20.52%</td>
<td>15.65%</td>
<td>45.65%</td>
</tr>
<tr>
<td>2</td>
<td>25.65%</td>
<td>19.56%</td>
<td>65.21%</td>
</tr>
<tr>
<td>3</td>
<td>21.38%</td>
<td>16.30%</td>
<td>81.51%</td>
</tr>
<tr>
<td>4</td>
<td>13.36%</td>
<td>10.19%</td>
<td>91.70%</td>
</tr>
<tr>
<td>5</td>
<td>6.68%</td>
<td>5.09%</td>
<td>96.80%</td>
</tr>
<tr>
<td>6</td>
<td>2.78%</td>
<td>2.12%</td>
<td>98.92%</td>
</tr>
<tr>
<td>7</td>
<td>0.99%</td>
<td>0.76%</td>
<td>99.68%</td>
</tr>
<tr>
<td>8</td>
<td>0.31%</td>
<td>0.24%</td>
<td>99.91%</td>
</tr>
</tbody>
</table>

Comment: We are given a 30% chance of zero claims. The remaining 70% is spread in proportion to the unmodified Poisson. For example, 
(70%)(20.52%)/(1-.0821) = 15.65%, and (70%)(25.65%)/(1-.0821) = 19.56%

Unlike the zero-truncated distribution, the zero-modified distribution has a probability of zero events.
14.18. A. The mode is where the density function is greatest, 0.

<table>
<thead>
<tr>
<th>Number of Claims</th>
<th>Unmodified Poisson</th>
<th>Zero-Modified Poisson</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.21%</td>
<td>30.00%</td>
</tr>
<tr>
<td>1</td>
<td>20.52%</td>
<td>15.65%</td>
</tr>
<tr>
<td>2</td>
<td>25.65%</td>
<td>19.56%</td>
</tr>
<tr>
<td>3</td>
<td>21.38%</td>
<td>16.30%</td>
</tr>
<tr>
<td>4</td>
<td>13.36%</td>
<td>10.19%</td>
</tr>
<tr>
<td>5</td>
<td>6.68%</td>
<td>5.09%</td>
</tr>
<tr>
<td>6</td>
<td>2.78%</td>
<td>2.12%</td>
</tr>
<tr>
<td>7</td>
<td>0.99%</td>
<td>0.76%</td>
</tr>
<tr>
<td>8</td>
<td>0.31%</td>
<td>0.24%</td>
</tr>
</tbody>
</table>

Comment: If the mode of the zero-modified and unmodified distribution are ≠ 0, then the zero-modified distribution has the same mode as the unmodified distribution, since all the densities on the positive integers are multiplied by the same factor.

14.19. A. \( \frac{f(x+1)}{f(x)} = \frac{2.4x!}{(x+1)!} = 2.4/(x+1) \). Thus this is a member of the (a, b, 0) subclass, \( \frac{f(x+1)}{f(x)} = a + \frac{b}{x+1} \), with \( a = 0 \) and \( b = 2.4 \). This is a Poisson Distribution, with \( \lambda = 2.4 \).

For the unmodified Poisson, the probability of more than zero claims is: \( 1 - e^{-2.4} \).

After, zero-modification, this probability is: \( 1 - .31 = .69 \). Thus the zero-modified distribution is, \( f_M(x) = \frac{.69/(1 - e^{-2.4})}{f(x)} = \frac{.69/(1 - e^{-2.4})}{e^{-2.4}} \frac{2.4^x}{x!} = \frac{2.4^x(.69)}{(e^{2.4} - 1) x!} \), \( x \geq 1 \).

\[ f_M(3) = 2.4^3(.69/(e^{2.4} - 1) 3!) = .159. \]

<table>
<thead>
<tr>
<th># claims</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>zero-modified density</td>
<td>0.31</td>
<td>0.1652</td>
<td>0.1983</td>
<td>0.1586</td>
<td>0.0952</td>
<td>0.0457</td>
<td>0.0183</td>
<td>0.0063</td>
</tr>
</tbody>
</table>

Comment: For a Poisson with \( \lambda = 2.4 \), \( f(n)/f(m) = (e^{-2.4} 2.4^n / n!)/(e^{-2.4} 2.4^m / m!) = 2.4^{n-m} m! / n! \).
14.20. Calculate \((x+1)f(x+1)/f(x) = (x+1)\) (number with x+1) / (number with x).

<table>
<thead>
<tr>
<th>Number of Days</th>
<th>Observed</th>
<th>((x+1)f(x+1)/f(x))</th>
<th>Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50,122</td>
<td>0.183</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>9,190</td>
<td>1.199</td>
<td>1.016</td>
</tr>
<tr>
<td>2</td>
<td>5,509</td>
<td>1.774</td>
<td>0.575</td>
</tr>
<tr>
<td>3</td>
<td>3,258</td>
<td>2.387</td>
<td>0.613</td>
</tr>
<tr>
<td>4</td>
<td>1,944</td>
<td>2.984</td>
<td>0.597</td>
</tr>
<tr>
<td>5</td>
<td>1,160</td>
<td>3.584</td>
<td>0.601</td>
</tr>
<tr>
<td>6</td>
<td>693</td>
<td>4.222</td>
<td>0.638</td>
</tr>
<tr>
<td>7</td>
<td>418</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8+</td>
<td>621</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The accident profile is not approximately linear starting at zero. Thus, this is probably not from a member of the \((a, b, 0)\) class. The accident profile is approximately linear starting at one. Thus, this is probably from a member of the \((a, b, 1)\) class.

Comment: \(f(x+1)/f(x) = a + b/(x+1)\), so \((x+1)f(x+1)/f(x) = a(x+1) + b = ax + a + b\). The slope is positive, so \(a > 0\) and we have a Negative Binomial. The slope, \(a \approx 0.6\). The intercept is about 0.6. Therefore, \(b \approx 0\).

For the Negative Binomial \(b = (r-1)\beta/(1+\beta)\). Thus \(b = 0\), implies \(r \approx 1\). Thus the data may have been drawn from a Zero-Modified Geometric, with \(\beta \approx 0.6\).

14.21. C. \(f(x+1)/ f(x) = x!/ (x+1)! = 1/(x+1)\). Thus this is a member of the \((a, b, 0)\) subclass, \(f(x+1)/ f(x) = a + b/(x+1)\), with \(a = 0\) and \(b = 1\). This is a Poisson Distribution, with \(\lambda = 1\).

For the unmodified Poisson, the probability of more than zero claims is: \(1 - e^{-1}\). After, zero-modification, this probability is: \(1 \cdot .1 = .9\). Thus the zero-modified distribution is,

\[
f_M(x) = (.9/(1-e^{-1}))f(x) = (.9/(1-e^{-1}))e^{-1} 1^x/x! = .9/((e - 1) x!), x \geq 1.
\]

\(f_M(1) = .9/(e-1) = .524\).

<table>
<thead>
<tr>
<th># claims</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.5238</td>
<td>0.2619</td>
<td>0.0873</td>
<td>0.0218</td>
<td>0.0044</td>
<td>0.0007</td>
<td></td>
</tr>
</tbody>
</table>

Comment: For a Poisson with \(\lambda = 1\), \(f(n)/f(m) = (e^{-1} 1^n / n!)/(e^{-1} 1^m / m!) = m! / n!\).

15.1. B. \(P(z) = P_1[P_2(z)]\).

The p.g.f of the primary Geometric is: \(1/(1 - \beta(z-1)) = 1/(1 - 1.7(z-1)) = 1/(2.7 - 1.7z)\).

The p.g.f of the secondary Poisson is: \(exp[\lambda(z-1)] = exp[3.1(z-1)]\).

Thus the p.g.f. of the compound distribution is \(1/(2.7 - 1.7exp[3.1(z-1)])\).
15.2. B. We are taking 1/3 of the claims from the secondary Binomial. Thus the secondary distribution is Binomial with \( q = \frac{.3}{3} = 0.1 \) and \( m = 3 \). Thus the frequency distribution of losses of size greater than $10,000 is given by a Poisson-Binomial compound frequency distribution, as per Loss Models with \( \lambda = .18, q = 0.1, \) and \( m = 3 \).

15.3. B. Provided the secondary distributions are the same, the primary distributions add as they usually would. The sum of two independent Binomials with the same \( q \), is another Binomial with the sum of the \( m \) parameters. In this case it is a Binomial with \( q = 0.15 \) and \( m = 3 + 5 = 8 \).\( X + Y \) is a Binomial-Geometric with \( q = 0.15, m = 8, \) and \( \beta = 2.3 \).

Comment: The secondary distributions determine how many claims there are per accident. The primary distributions determine how many accidents. In this case the Binomial distributions of the number of accidents add.

15.4. E. \( P(z) = P_1[P_2(z)] \).

Density at 0 is: \( P(0) = P_1[P_2(0)] = P_1[e^{-0.7}] = .2e^{-0.7} + .5e^{-1.4} + .3e^{-2.1} = .259 \).

Alternately, the primary distribution has 20% probability of 1, 50% probability of 2, and 30% probability of 3, while the secondary distribution is a Poisson with \( \lambda = 0.7 \).

The density at zero of the secondary distribution is \( e^{-0.7} \).

Therefore, the probability of zero claims for the compound distribution is:

\[
(0.2)(\text{Prob } 0 \text{ from secondary}) + (0.5)(\text{Prob } 0 \text{ from secondary})^2 + (0.3)(\text{Prob } 0 \text{ from secondary})^3
\]

\[= .2e^{-0.7} + .5(e^{-0.7})^2 + .3(e^{-0.7})^3 = .259.\]

15.5. C. One adds up 20,000 independent identically distributed variables. In the case of a Compound Poisson distribution, the primary Poissons add to give another Poisson with \( \lambda_1 = (20000)(.03) = 600 \). The secondary distribution stays the same.

The portfolio has a compound Poisson-Poisson with \( \lambda_1 = 600 \) and \( \lambda_2 = .07 \).

15.6. C. The p.g.f of the primary Poisson is \( \exp(\lambda(z-1)) = \exp(1.2(z-1)) \).

The p.g.f of the secondary Binomial is \( \{1 + q(z-1)\}^m = \{1 + .1(z-1)\}^4 \).

Thus the p.g.f of the compound distribution is \( P(z) = P_1[P_2(z)] = \exp(1.2(\{1 + .1(z-1)\}^4-1)) \).

15.7. B. Poisson-Geometric with \( \lambda = \frac{75}{100}(4) = 3 \) and \( \beta = 0.8 \).

Comment: One adjusts the primary Poisson distribution in a manner similar to that if one just had a Poisson distribution.
15.8. D. \( P(z) = P_1[P_2(z)] \).
Density at 0 is: \( P(0) = P_1[P_2(0)] = P_1[.8^3] = 1/\{5 - 4(.8^3)\} = .339 \).
Alternately, the secondary distribution is a Binomial with \( m = 3 \) and \( q = 0.2 \).
The density at zero of the secondary distribution is \( .8^3 \).
Therefore, the probability of zero claims for the compound distribution is:
\[ P_1[.8^3] = 1/\{5 - 4(.8^3)\} = .339 \.

15.9. A. Negative Binomial-Poisson with \( \beta = 0.4, r = (500/50)(3) = 30, \) and \( \lambda = 0.7 \).
Comment: One adjusts the primary Negative Binomial distribution in a manner similar to that if one just had a Negative Binomial distribution.

15.10. E. The p.g.f. of the compound distribution is the p.g.f. of the primary distribution at the p.g.f. of the secondary distribution: \( P(z) = [1 - 3(\exp[\lambda(z - 1)] - 1)]^{-2} \).
\[ 0.067 = f(0) = P(0) = [1 - 3(\exp[\lambda(0 - 1)] - 1)]^{-2} = [1 - 3(\exp[-\lambda] - 1)]^{-2}. \]
\[ \Rightarrow 1 - 3(\exp[-\lambda] - 1) = 3.8633. \Rightarrow \exp[-\lambda] = .04555. \Rightarrow \lambda = 3.089. \]
Alternately, the Poisson secondary distribution at zero is \( e^{-\lambda} \).
From the first step of the Panjer Algorithm, \( c(0) = P_p[s(0)] = [1 - 3(e^{-\lambda} - 1)]^{-2} \). Proceed as before.
Comment: \( P(z) = E[z^n] = \sum f(n)z^n \). Therefore, letting \( z \) approach zero, \( P(0) = f(0) \).

16.1. E. Standard deviation of the compound distribution is:
\[ \sqrt{(15)(4^2) + (10^2)(3^2)} = \sqrt{1140} = 33.8. \]

16.2. E. The frequency over a year is Poisson with mean: \( (12)(10) = 120 \) accidents.
Second moment of the secondary distribution is: \( (40\%)(1^2) + (40\%)(2^2) + (20\%)(3^2) = 3.8 \).
Variance of compound distribution is: \( (120)(3.8) = 456 \).
Comment: Similar to 3, 5/01, Q.36.

16.3. B. The mean of the primary Geometric Distribution is 1.7. The mean of the secondary Poisson Distribution is 3.1. Thus the mean of the compound distribution is: \( (1.7)(3.1) = 5.27 \).
16.4. A. The mean of the primary Geometric Distribution is 1.7. The mean of the secondary Poisson Distribution is 3.1. The variance of the primary Geometric is: \((1.7)(1+1.7) = 4.59\). The variance of the secondary Poisson Distribution is 3.1.

The variance of the compound distribution is: \((1.7)(3.1) + (3.1)^2(4.59) = 49.38\).

**Comment:** The variance of the compound distribution is large compared to its mean. A very large number of items can result if there are a large number of customers from the Geometric combined with some of those customers buying a large numbers of items from the Poisson. Compound distributions tend to have relatively heavy tails.

16.5. E. From the previous solutions, the mean of the compound distribution is 5.27, and the variance of the compound distribution is 49.38. Thus the standard deviation is 7.03.

\[
1 - \Phi((4.5 - 5.27)/7.03) = 1 - \Phi(-.11) = \Phi(.11) = .5438.
\]

16.6. C. Over \(y\) hours, the number of salmon has mean 50\(y\) and variance 100\(y\).

The mean aggregate number of eggs is: \((50y)(3000) = 150000y\).

The standard deviation of the aggregate number of eggs is:

\[
\sqrt{(50y)(1000^2) + (3000^2)(100y)} = 30822\sqrt{y}.
\]

Thus the probability that the aggregate number of eggs is < 2 million is approximately:

\[
\Phi((1999999.5 - 150000y)/30822\sqrt{y}).
\]

Since \(\Phi(2.576) = .995\), this probability will be 1/2% if:

\[
(1999999.5 - 150000y)/30822\sqrt{y} = -2.576 \Rightarrow 150000y - 79397\sqrt{y} - 1999999.5 = 0.
\]

\[
\sqrt{y} = (79397 \pm \sqrt{79397^2 + 4(150000)(1999999.5)})/ (2(150000)) = .2647 \pm 3.6611.
\]

\[
\sqrt{y} = 3.926. \ y = 15.4. \ The \ smallest \ whole \ number \ of \ hours \ is \ therefore \ 16.
\]

Alternately, try the given choices and stop when \((\text{Mean} - 2\text{ million})/\text{StdDev.} > 2.576\).

<table>
<thead>
<tr>
<th>Hours</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th># of Claims</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>2,100,000</td>
<td>115,325</td>
<td>0.867</td>
</tr>
<tr>
<td>15</td>
<td>2,250,000</td>
<td>119,373</td>
<td>2.094</td>
</tr>
<tr>
<td>16</td>
<td>2,400,000</td>
<td>123,288</td>
<td>3.244</td>
</tr>
<tr>
<td>17</td>
<td>2,550,000</td>
<td>127,082</td>
<td>4.328</td>
</tr>
<tr>
<td>18</td>
<td>2,700,000</td>
<td>130,767</td>
<td>5.353</td>
</tr>
</tbody>
</table>

**Comment:** Similar to 3, 5/01, Q.16.

Note that since the variance over one hour is 100, the variance of the number of salmon over two hours is: \((2)(100) = 200\).

Number of salmon over two hours = number over the first hour + number over the second hour.

\[
\Rightarrow \text{Var}[\text{Number over two hours}] = \text{Var}[\text{number over first hour}] + \text{Var}[\text{number over second hour}]
\]

\[
= 2 \text{ Var}[\text{number over an hour}]. \ We \ are \ adding \ independent \ random \ variables, \ rather \ than \ multiplying \ an \ individual \ variable \ by \ a \ constant.
\]
16.7. B. The mean frequency over 5 days is: \(7(5) = 35\).
Mean number of claims per envelope is: \((60\%)(1) + (30\%)(2) + (10\%)(3) = 1.5\).
Mean of compound distribution is: \((35)(1.5) = 52.5\).
Second moment of number of claims per envelope is: 
\[\text{Second moment} = (60\%)(1^2) + (30\%)(2^2) + (10\%)(3^2) = 2.7.\]
Variance of compound distribution is: \((35)(2.7) = 94.5\).
99th percentile \(\cong \text{mean} + (2.326)(\text{standard deviations}) = 52.5 + (2.326)\sqrt{94.5} = 75.1\).
Comment: Similar to 3, 11/01, Q.30.

16.8. C. The number of persons has mean: \((13)(2) = 26\),
and variance: \((13)(2 + 1) = 78\).
The number of transactions per person has mean:
\(\text{(30\%)(1) + (40\%)(2) + (20\%)(3) + (10\%)(4) = 2.1,}\)
second moment: 
\[\text{Second moment} = (30\%)(1^2) + (40\%)(2^2) + (20\%)(3^2) + (10\%)(4^2) = 5.3,\]
and variance: \(5.3 - 2.1^2 = .89.\)
The number of transactions in an hour has mean: \((26)(2.1) = 54.6,\)
and variance: \((26)(.89) + (2.1^2)(78) = 367.12.\)
The number of transactions in 5 hours has mean: \((5)(54.6) = 273,\)
and variance: \((5)(367.12) = 1835.6.\)
\(\Phi(.842) = 80\%. \) 80th percentile \(\cong 273 + (.842)\sqrt{1835.6} = 309.1.\)

16.9. E. Mean of the Primary Negative Binomial = \((100)(0.6) = 60.\)
Variance of the Primary Negative Binomial = \((100)(0.6)(1.6) = 96.\)
Mean of the Secondary Binomial = \((4)(0.1) = .4.\)
Variance of the Secondary Binomial = \((4)(0.1)(0.9) = .36.\)
Variance of the Compound Distribution = \((60)(.36) + (.4^2)(96) = 36.96.\)

16.10. D. Mean of the Compound Distribution = \((60) (.4) = 24.\)
\(\text{Prob[# claimants} \leq 20] = \Phi((20.5 - 24)/\sqrt{36.96}) = \Phi(-.58) = 1 -.7190 = 28.1\%.\)

Mean Severity: \((3)(4000) = 12,000.\)  Variance of Severity: \((3)(4000^2) = 48,000,000.\)
Mean Aggregate Loss = \((24)(12000) = 288,000.\)
Variance of the Aggregate Loss = \((24)(48,000,000) + (12,000^2)(36.96) = 6474 \text{ million.}\)
\(\text{Prob[Aggregate loss} > 300000] = 1 - \Phi((300000 - 288000)/\sqrt{6474 \text{ million}}) = \)
\(1 - \Phi(.15) = 1 - .5596 = 44\%.\)
16.12. E. The number of batters has mean: $3 + (1.4)(1) = 4.4$, and variance: $(1.4)(1)(1 + 1) = 2.8$.
The number of pitches per batter has mean: $1 + (1.8)(1.5) = 3.7$,
and variance: $(1.8)(1.5)(1 + 1.5) = 6.75$.
The number of pitches per half-inning has mean: $(4.4)(3.7) = 16.28$,
and variance: $(4.4)(6.75) + (3.7^2)(2.8) = 68.032$.
Prob[# pitches > 30] $\approx 1 - \Phi([30.5 - 16.28]/\sqrt{68.032}] = 1 - \Phi(1.72) = 4.27\%$.

16.13. D. Over y minutes, the number of taxicabs has mean 5.6y and variance 5.6y.
The passengers per cab has mean: $(.3)(4) = 1.2$, and variance: $(.3)(1 - .3)(4) = .84$.
The mean aggregate number of passengers is: $(5.6y)(1.2) = 6.72y$.
The standard deviation of the aggregate number of passengers is:
$\sqrt{(5.6y)(.84) + (1.2^2)(5.6y)} = 3.573\sqrt{y}$.
Thus the probability that the aggregate number of passengers is ≥ 1000 is approximately:
$1 - \Phi((999.5 - 6.72y)/3.573\sqrt{y})$. Since $\Phi(1.282) = .90$, this probability will be greater than 90% if:
(Mean - 999.5)/StdDev. = $(6.72y - 999.5)/3.573\sqrt{y} > 1.282$.
Try the given choices and stop when (Mean - 999.5)/StdDev. > 1.282.

<table>
<thead>
<tr>
<th>Minutes</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th># of Claims</th>
</tr>
</thead>
<tbody>
<tr>
<td>155</td>
<td>1,041.6</td>
<td>44.48</td>
<td>0.946</td>
</tr>
<tr>
<td>156</td>
<td>1,048.3</td>
<td>44.63</td>
<td>1.094</td>
</tr>
<tr>
<td>157</td>
<td>1,055.0</td>
<td>44.77</td>
<td>1.241</td>
</tr>
<tr>
<td>158</td>
<td>1,061.8</td>
<td>44.91</td>
<td>1.386</td>
</tr>
<tr>
<td>159</td>
<td>1,068.5</td>
<td>45.05</td>
<td>1.531</td>
</tr>
</tbody>
</table>

The smallest whole number of minutes is therefore 158.

16.14. A. The mean number of lawyers is: 2.5 and the variance is:
$\{(1 - 2.5)^2 + (2 - 2.5)^2 + (3 - 2.5)^2 + (4 - 2.5)^2\}/4 = 1.25$.
The mean number of hours per lawyer is: $(7)(.6) = 4.2$ and the variance is: $(7)(.4)(.6) = 1.68$.
Therefore, the total number of hours volunteered per day has mean: $(2.5)(4.2) = 10.5$ and variance:
$(2.5)(1.68) + (4.2^2)(1.25) = 26.25$.
The number of clients per hour has mean 5 and variance 5.
Therefore, the total number of clients per day has mean: $(5)(10.5) = 52.5$,
and variance: $(10.5)(5) + (5^2)(26.25) = 708.75$.
Prob[# clients ≥ 40] $\equiv 1 - \Phi((39.5 - 52.5)/\sqrt{708.75}) = 1 - \Phi(-.49) = 68.79\%$.
Alternately, the mean number of clients per lawyer is: $(4.2)(5) = 21$
with variance: $(4.2)(5) + (5^2)(1.68) = 63$.
Therefore, the total number of clients per day has mean: $(2.5)(21) = 52.5$ and variance: $(2.5)(63) + (21^2)(1.25) = 708.75$. Proceed as before.
Comment: Similar to 3, 11/00, Q.2.
16.15. A. The mean of the primary Poisson Distribution is 1.2.
The mean of the secondary Binomial Distribution is: \(4\)(.1) = .4.
Thus the mean of the compound distribution is: \((1.2)(.4) = .48\).

16.16. B. The mean of the primary Poisson Distribution is 1.2. The mean of the secondary Binomial Distribution is: \(4\)(.1) = .4. The variance of the primary Poisson Distribution is 1.2.
The variance of the secondary Binomial Distribution is: \((4)(.1)(.9) = .36\).
The variance of the compound distribution is: \((1.2)(.36) + (.4)^2(1.2) = .624\).

16.17. A. The compound distribution has mean of .48 and variance of .624.
Prob\[# books > 2\] \(\cong 1 - \Phi((2.5 - .48)/\sqrt{.624}) = 1 - \Phi(2.56) = 1 - .9948 = .0052\).

16.18. B. The mean number of nuggets per pan is: \((90\%)(1) + (9\%)(5) + (1\%)(25) = 1.6\).
The 2nd moment of the number of nuggets per pan is: \((90\%)(1^2) + (9\%)(5^2) + (1\%)(25^2) = 9.4\).
Mean aggregate over 10 days is: \((10)(3)(1.6) = 48\).
Variance of aggregate over 10 days is: \((10)(3)(9.4) = 282\).
Prob[aggregate < 30] \(\cong \Phi((29.5 - 48)/\sqrt{282}) = \Phi(-1.10) = 13.57\%\).

16.19. A. This is a compound frequency distribution with a primary distribution that is discrete and uniform on 1 through 5 and with secondary distribution which is Poisson with \(\lambda = 30\). The primary distribution has mean of 3 and second moment of:
\((1^2 + 2^2 + 3^2 + 4^2 + 5^2)/5 = 11\). Thus the primary distribution has variance: \(11 - 3^2 = 2\).
Mean of the Compound Dist. = (Mean of Primary Dist.)\(\times\) (Mean of Secondary Dist.) = \((3)(30) = 90\).
Variance of the Compound Distribution =
(Mean of Primary Dist.)(Variance of Secondary Dist.) +
(Mean of Secondary Dist.)^2\(\times\) (Variance of Primary Dist.) = \((3)(30) + (30^2)(2) = 1890\).
Probability of 120 or more patients \(\cong 1 - \Phi((119.5 - 90)/\sqrt{1890}) = 1 - \Phi(0.68)\).
16.20. B. Over $y$ hours, the number of salmon has mean $100y$ and variance $900y$. The mean aggregate number of eggs is: $(100y)(5) = 500y$. The variance of the aggregate number of eggs is: $$(100y)(5) + (5^2)(900y) = 23000y.$$ Thus the probability that the aggregate number of eggs is < 10000 is approximately: $$\Phi\left(\frac{9999.5 - 500y}{\sqrt{23000y}}\right).$$ Since $\Phi(1.645) = .95$, this probability will be 5% if: $$(9999.5 - 500y)/\sqrt{23000y} = -1.645 \Rightarrow 500y - 249.98\sqrt{y} - 9999.5 = 0.$$ $$\sqrt{y} = 4.729. \ y = 22.3. \ \text{The smallest whole number of hours is therefore 23.}$$

Alternately, calculate the probability for each of the number of hours in the choices.

<table>
<thead>
<tr>
<th>Hours</th>
<th>Mean</th>
<th>Variance</th>
<th>Probability of at least 10,000 eggs</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>10,000</td>
<td>460,000</td>
<td>$1 - \Phi((9999.5 - 10000)/\sqrt{460000}) = 1 - \Phi(-0.007) = 50.0%$</td>
</tr>
<tr>
<td>23</td>
<td>11,500</td>
<td>529,000</td>
<td>$1 - \Phi((9999.5 - 11500)/\sqrt{529000}) = 1 - \Phi(-2.063) = 98.0%$</td>
</tr>
<tr>
<td>26</td>
<td>13,000</td>
<td>598,000</td>
<td>$1 - \Phi((9999.5 - 13000)/\sqrt{598000}) = 1 - \Phi(-3.880) = 99.995%$</td>
</tr>
</tbody>
</table>

Thus 20 hours is not enough and 23 hours is enough so that the probability is greater than 95%.

Comment: The number of salmon acts as the primary distribution, and the number of eggs per salmon as the secondary distribution. This exam question should have been worded better. They intended to say “so the probability that at least 10,000 eggs will be released is greater than 95%.” The probability of exactly 10,000 eggs being released is very small.

16.21. E. The second moment of the number of claimants per accident is: $$(1/2)(1^2) + (1/3)(2^2) + (1/6)(3^2) = 3.333. \ \text{The variance of a Compound Poisson Distribution is:} \ 
\lambda(\text{2nd moment of the secondary distribution}) = (12)(3.333) = 40.$$ Alternately, thinning the original Poisson, those accidents with 1, 2, or 3 claimants are independent Poissons. Their means are: $(1/2)(12) = 6$, $(1/3)(12) = 4$, and $(1/6)(12) = 2$.

Number of accidents with 3 claimants is Poisson with mean 2 $\Rightarrow$ The variance of the number of accidents with 3 claimants is 2.

Number of claimants for those accidents with 3 claimants = $(3)(\text{# of accidents with 3 claimants}) \Rightarrow$ The variance of the # of claimants for those accidents with 3 claimants is: $(3^2)(2)$.

Due to independence, the variances of the three processes add: $(1^2)(6) + (2^2)(4) + (3^2)(2) = 40$.

16.22. B. Mean # claims / envelope = $(1)(.2) + (2)(.25) + (3)(.4) + (4)(.15) = 2.5.$
2nd moment # claims / envelope = $(1^2)(.2) + (2^2)(.25) + (3^2)(.4) + (4^2)(.15) = 7.2.$
Over 13 weeks, the number of envelopes is Poisson with mean: $(13)(50) = 650.$
Mean of the compound distribution = $(650)(2.5) = 1625.$
Variance of the aggregate number of claims = Variance of a compound Poisson distribution = (mean primary Poisson distribution)(2nd moment of the secondary distribution) = $(650)(7.2) = 4680.$ $\Phi(1.282) = .90.$ Estimated 90th percentile $= 1625 + 1.282 \sqrt{4680} = 1713.$
16.23. E. The amount won per a round of the game is a compound frequency distribution. Primary distribution (determining how many dice are rolled) is a six-sided die, uniform and discrete on 1 through 6, with mean 3.5, second moment \((1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2)/6 = 91/6\) and variance \(91/6 - 3.5^2 = 35/12\).

Secondary distribution is also a six-sided die, with mean 3.5 and variance 35/12. Mean of the compound distribution is: \((3.5)(3.5) = 12.25\).

Variance of the compound distribution is: \((3.5)(35/12) + (3.5^2)(35/12) = 45.94\).

Therefore, the net result of a round has mean \(12.25 - 12.5 = -0.25\), and variance 45.94. 1000 rounds have a net result with mean -250 and variance 45,940.

\[\text{Prob[net result } \geq 0] \approx 1 - \Phi((-0.5 + 250)/\sqrt{45,940}) = 1 - \Phi(1.16) = 1 - .8770 = .1220.\]

16.24. B. The total number of delayed passengers is a compound frequency distribution, with primary distribution the number of delayed flights, and the secondary distribution the number of passengers on a flight. The number of flights delayed per year is Poisson with mean: \((2)(12) = 24\).

The second moment of the secondary distribution is: \(50^2 + 30^2 = 3400\).

The variance of the number of passengers delayed per year is: \((24)(3400) = 81,600\).

The standard deviation of the number of passengers delayed per year is: \(\sqrt{81,600} = 285.66\).

The standard deviation of the annual compensation is: \((100)(285.66) = 28,566\).

16.25. D. The mean number of sessions is: \((400)(.2)(2) + (300)(.5)(15) + (200)(.3)(9) = 2950\).

For a single resident we have a Bernoulli primary (whether the resident need therapy) and a geometric secondary (how many visits).

This has variance: \((\text{mean of primary})(\text{variance of second.}) + (\text{mean second.})^2(\text{var. of primary}) = q\beta(1 + \beta) + \beta^2q(1 - q)\).

For a resident in state 1, the variance of the number of visits is: \(.2)(2)(3) + (3^2)(.2)(1 - .8) = 1.84\).

For state 2, the variance of the number of visits is: \(.5)(15)(16) + (15^2)(.5)(1 - .5) = 176.25\).

For state 3, the variance of the number of visits is: \(.3)(9)(10) + (9^2)(.3)(1 - .3) = 44.01\).

The sum of the visits from 400 residents in state 1, 300 in state 2, and 200 in state 3, has variance: \((400)(1.84) + (300)(176.25) + (200)(44.01) = 62,413\).

\[\text{Prob[sessions } > 3000] \approx 1 - \Phi((3000.5 - 2950)/\sqrt{62413}) = 1 - \Phi[.20] = .4207.\]

16.26. E. Primary distribution has mean: \((0)(.1) + (1)(.4) + (2)(.3) + (3)(.2) = 1.6\), second moment: \((0^2)(.1) + (1^2)(.4) + (2^2)(.3) + (3^2)(.2) = 3.4\), and variance: \(3.4 - 1.6^2 = 0.84\).

The secondary distribution has mean 3 and variance 3.

The compound distribution has variance: \((1.6)(3) + (3^2)(0.84) = 12.36\).
16.27. E. Mean = (mean primary)(mean secondary) = (100)(1.1)(1.0) = 110.

Variance = (mean primary)(variance of secondary) + (mean secondary)^2(variance of primary) = (100)(1.1)(1)(1 + 1) + ((1.1)(1.0))^2(100) = 341. \( \Phi(2.326) = 0.99. \)

99th percentile: 110 + 2.326\sqrt{341} = 152.95. Need at least 153 televisions.

16.28. A. The primary distribution is Binomial with \( m = 1000 \) and \( q = .2 \), with mean 200 and variance 160. The mean of the compound distribution is: \( (200)(20) = 4000. \)

The variance of the compound distribution is: \( (200)(20) + (20^2)(160) = 68,000. \)

Annual budget is: \( 10(4000 + \sqrt{68000}) = 42,608. \)

Solutions for problems in the remaining sections are in Study Aid I.