### Chapter 3: Survival Distributions and Life Tables

**Distribution function of** $X$:

$$F_X(x) = \Pr(X \leq x)$$

**Survival function** $s(x)$:

$$s(x) = 1 - F_X(x)$$

**Probability of death between age $x$ and age $y$**:

$$\Pr(x < X \leq z) = F_X(z) - F_X(x) = s(x) - s(z)$$

**Probability of death between age $x$ and age $y$ given survival to age $x$**:

$$\Pr(x < X \leq z|X > x) = \frac{F_X(z) - F_X(x)}{1 - F_X(x)} = \frac{s(x) - s(z)}{s(x)}$$

**Notations**:

- $tq_x = \Pr[T(x) \leq t]$
- $tp_x = \Pr[T(x) > t]$
- $t|u q_x = \Pr[t < T(x) \leq t + u]$

**Relations with survival functions**:

$$tp_x = \frac{s(x + t)}{s(x)}$$

$$tq_x = 1 - \frac{s(x + t)}{s(x)}$$

**Curtate future lifetime ($K(x) \equiv$ greatest integer in $T(x)$)**:

$$\Pr[K(x) = k] = \Pr[k \leq T(x) < k + 1] = kp_x - k+1p_x = kp_x \cdot q_{x+k} = k\cdot q_x$$

**Force of mortality** $\mu(x)$:

$$\mu(x) = \frac{f_X(x)}{1 - F_X(x)} = \frac{s'(x)}{s(x)}$$

**Relations between survival functions and force of mortality**:

$$s(x) = \exp \left( - \int_0^x \mu(y) \, dy \right)$$

$$np_x = \exp \left( - \int_x^{x+n} \mu(y) \, dy \right)$$

**Derivatives**:

$$\frac{d}{dt} tp_x = tp_x \cdot \mu(x + t) = f_T(x)(t)$$

$$\frac{d}{dt} t q_x = -t p_x \cdot \mu(x + t)$$

$$\frac{d}{dt} T_x = -l_x$$

$$\frac{d}{dt} L_x = -d_x$$

$$\frac{d}{dt} \hat{e}_x = \mu(x)\hat{e}_x - 1$$

**Mean and variance of $T$ and $K$**:

$$E[T(x)] = \text{complete expectation of life}$$

$$\hat{e}_x = \int_0^\infty t p_x \, dt$$

$$E[K(x)] = \text{curtate expectation of life}$$

$$e_x = \sum_{k=1}^{\infty} kp_x$$

$$\text{Var}[T(x)] = 2 \int_0^\infty t \cdot t p_x \, dt - \hat{e}_x^2$$

$$\text{Var}[K(x)] = \sum_{k=1}^{\infty} (2k - 1) kp_x - e_x^2$$

**Total lifetime after age $x$: $T_x$**

$$T_x = \int_0^\infty l_{x+t} \, dt$$
Total lifetime between age $x$ and $x+1$: $L_x$

$$L_x = T_x - T_{x+1} = \int_0^1 l_{x+t} \, dt = \int_0^1 l_x \cdot t p_x \, dt$$

Total lifetime from age $x$ to $x+n$: $nL_x$

$$nL_x = T_x - T_{x+n} = \sum_{k=0}^{n-1} L_{x+k} = \int_0^n l_{x+t} \, dt$$

Average lifetime after $x$: $\hat{e}_x$

$$\hat{e}_x = \frac{T_x}{l_x}$$

Average lifetime from $x$ to $x+1$: $\hat{e}_{x:1}$

$$\hat{e}_{x:1} = \frac{L_x}{l_x}$$

Median future lifetime of $(x)$: $m(x)$

$$Pr[T(x) > m(x)] = \frac{s(x + m(x))}{s(x)} = \frac{1}{2}$$

Central death rate: $m_x$

$$m_x = \frac{l_x - l_{x+1}}{L_x}$$

$$n m_x = \frac{l_x - l_{x+n}}{n L_x}$$

Fraction of year lived between age $x$ and age $x+1$ by $d_x$: $a(x)$

$$a(x) = \frac{1}{\int_0^1 t \cdot t p_x \cdot \mu(x + t) \, dt} = \frac{L_x - l_{x+1}}{l_x - l_{x+1}}$$

Recursion formulas:

$$E[K] = e_x = p_x (1 + e_{x+1})$$

$$E[T] = \hat{e}_x = p_x (1 + \hat{e}_{x+1}) + q_x a(x)$$

$$e_x = e_{x:m} + n p_x e_{x+n}$$

$$\hat{e}_x = \hat{e}_{x:m} + n p_x \hat{e}_{x+n}$$

$$E[K \land (m+n)] = e_{x:m+n}$$

$$E[T \land (m+n)] = \hat{e}_{x:m+n}$$

$$E[K \land (m+n)] = e_{x:m+n} + m p_x e_{x+m:n}$$

$$E[T \land (m+n)] = \hat{e}_{x:m+n} + m p_x \hat{e}_{x+m:n}$$
Chapter 4: Life Insurance

Whole life insurance: $\bar{A}_x$

$$E[Z] = \bar{A}_x = \int_0^\infty v^t \cdot t p_x \mu_x(t) dt$$
$$\text{Var}[Z] = 2\bar{A}_x - (\bar{A}_x)^2$$

$n$-year term insurance: $\bar{A}_{x:n}$

$$E[Z] = \bar{A}_{x:n} = \int_0^n v^t \cdot t p_x \mu_x(t) dt$$
$$\text{Var}[Z] = 2\bar{A}_{x:n} - (\bar{A}_{x:n})^2$$

$m$-year deferred whole life: $m|\bar{A}_x$

$$E[Z] = m|\bar{A}_x = \int_0^m v^t \cdot t p_x \mu_x(t) dt$$
$$m|\bar{A}_x = \bar{A}_x - \bar{A}_{x:m}$$

$n$-year pure endowment: $\bar{A}_{x:1}$

$$E[Z] = \bar{A}_{x:1} = n \cdot p_x = n E_x$$
$$\text{Var}[Z] = 2\bar{A}_{x:1} - (\bar{A}_{x:1})^2 = v^{2n} \cdot n p_x \cdot n q_x$$

$n$-year endowment insurance: $\bar{A}_{x:n}$

$$E[Z] = \bar{A}_{x:n} = \int_0^n v^t \cdot t p_x \mu_x(t) dt + v^n \cdot n p_x$$
$$\text{Var}[Z] = 2\bar{A}_{x:n} - (\bar{A}_{x:n})^2$$
$$\bar{A}_{x:n} = \bar{A}_{x:1} + \bar{A}_{x:1}$$

$m$-yr deferred $n$-yr term: $m|n \bar{A}_x$

$$m|n \bar{A}_x = m E_x \cdot \bar{A}_{x+i:n:m}$$
$$= \bar{A}_{x+m:n} - \bar{A}_{x:n}$$
$$= m|\bar{A}_x - m+n|\bar{A}_x$$

Discrete whole life: $A_x$

$$E[Z] = A_x = \sum_{k=0}^\infty v^{k+1} \cdot k p_x \cdot q_{x+k}$$
$$\text{Var}[Z] = 2A_x - (A_x)^2$$

Discrete $n$-year term: $A_{x:n}$

$$E[Z] = A_{x:n} = \sum_{k=0}^{n-1} v^{k+1} \cdot k p_x \cdot q_{x+k}$$
$$\text{Var}[Z] = 2A_{x:n} - (A_{x:n})^2$$

Discrete $n$-year endowment: $A_{x:n}$

$$E[Z] = A_{x:n} = \sum_{k=0}^{n-1} v^{k+1} \cdot k p_x \cdot q_{x+k} + v^n \cdot n p_x$$
$$\text{Var}[Z] = 2A_{x:n} - (A_{x:n})^2$$

$A_{x:n} = A_{x:1} + A_{x:1}$

Recursion and other relations:

$$\bar{A}_x = \bar{A}_{x:m} + n|\bar{A}_x$$
$$2\bar{A}_x = 2\bar{A}_{x:m} + n^2\bar{A}_x$$
$$n|\bar{A}_x = n E_x \cdot \bar{A}_{x+n}$$
$$\bar{A}_x = \bar{A}_{x:m} + n E_x \cdot \bar{A}_{x+n}$$
$$A_x = v q_x + v p_x A_{x+1}$$
$$2A_x = v^2 q_x + v^2 p_x A_{x+1}$$
$$A_{x:n} = v q_x + v p_x A_{x+1:n}$$
$$m|n A_x = v p_x \cdot (m-1)[n A_{x+1}]$$
$$A_{x:n} = v$$
$$A_{x:2} = v q_x + v^2 p_x$$
Varying benefit insurances:

\[
(I\bar{A})_x = \int_0^\infty [t + 1]v^t \cdot t p_x \mu_x(t) dt
\]

\[
(IA)_{x:m} = \int_0^m [t + 1]v^t \cdot t p_x \mu_x(t) dt
\]

\[
(TA)_x = \int_0^\infty t \cdot v^t \cdot t p_x \mu_x(t) dt
\]

\[
(T\bar{A})_{x:m} = \int_0^m t \cdot v^t \cdot t p_x \mu_x(t) dt
\]

\[
(IA)_{x:m} = \int_0^m (n - [t])v^t \cdot t p_x \mu_x(t) dt
\]

\[
(D\bar{A})_{x:m} = \int_0^m (n - t)v^t \cdot t p_x \mu_x(t) dt
\]

\[
(IA)_x = A_x + vp_x(IA)_{x+1}
\]

\[
= nv_x + vp_x[(IA)_{x+1} + A_{x+1}]
\]

\[
(IA)_{x:m} = nA_x + vp_x(IA)_{x+1}\]

\[
+ (IA)_{x:m} = [n(IA)_{x+1} + A_{x+1}]
\]

\[
(IA)_{x:m} = (n + 1)A_x_{x+1}\]

\[
(IA)_{x:m} = (n + 1)A_x_{x+1}
\]

Accumulated cost of insurance:

\[
n\bar{k}_x = \frac{A_x_{x:m}}{nE_x}
\]

Share of the survivor:

accumulation factor = \frac{1}{nE_x} = \frac{(1 + i)^n}{np_x}

Interest theory reminder

\[
a_m = \frac{1 - v^n}{i}
\]

\[
\bar{a}_m = \frac{1 - v^n}{\delta} = \frac{i}{\delta} a_m
\]

\[
\bar{a}_\infty = \frac{1}{\delta}, \quad a_\infty = \frac{1}{i}, \quad \bar{a}_\infty = \frac{1}{d}
\]

\[
(Ia)_m = \frac{\bar{a}_m - n v^n}{i}
\]

\[
(Da)_m = \frac{n - \bar{a}_m}{\delta}
\]

\[
(n + 1)a_m = (Ia)_m + (Da)_m
\]

\[
\bar{s}_m = \frac{i}{\delta}
\]

\[
d = iv
\]

\[
(Ia)_\infty = \frac{1}{id} = \frac{1 + i}{i^2}
\]

Doubling the constant force of interest \( \delta \)

1 + i \rightarrow (1 + i)^2

\[
v \rightarrow v^2
\]

\[
i \rightarrow 2i + i^2
\]

\[
d \rightarrow 2d - d^2
\]

\[
i \rightarrow \frac{2i + i^2}{2d}
\]

Limit of interest rate \( i = 0 \):

\[
A_x \overset{i=0}{\longrightarrow} 1
\]

\[
A_{x:m} \overset{i=0}{\longrightarrow} nq_x
\]

\[
n|A_x \overset{i=0}{\longrightarrow} np_x
\]

\[
A_{x:m} \overset{i=0}{\longrightarrow} 1
\]

\[
m|nA_x \overset{i=0}{\longrightarrow} mn|nq_x
\]

\[
(IA)_x \overset{i=0}{\longrightarrow} 1 + e_x
\]

\[
(TA)_x \overset{i=0}{\longrightarrow} \bar{e}_x
\]
Chapter 5: Life Annuities

Whole life annuity: $\ddot{a}_x$

$$\ddot{a}_x = E[\ddot{a}_x] = \int_0^\infty \ddot{a}_t \cdot t p_x \mu(x + t) dt$$

$$= \int_0^\infty v^t \cdot t p_x dt = \int_0^\infty t E_x dt$$

$$\text{Var}[\ddot{a}_x] = \frac{2\ddot{A}_x - (\dddot{A}_x)^2}{\delta^2}$$

$n$-year temporary annuity: $\ddot{a}_{x:n}$

$$\ddot{a}_{x:n} = \int_0^n v^t \cdot t p_x dt = \int_0^n t E_x dt$$

$$\text{Var}[Y] = \frac{2\ddot{A}_{x:n} - (\dddot{A}_{x:n})^2}{\delta^2}$$

$n$-year deferred annuity: $\ddot{a}_{x:n}$

$$\ddot{a}_{x:n} = \int_0^\infty v^t \cdot t p_x dt = \int_0^\infty t E_x dt$$

$$\text{Var}[Y] = \frac{2\ddot{A}_{x:n} - (\dddot{A}_{x:n})^2}{\delta^2}$$

$n$-yr certain and life annuity: $\ddot{a}_{x:n}$

$$\ddot{a}_{x:n} = \ddot{a}_x + \ddot{a}_m - \ddot{a}_{x:n}$$

$$= \ddot{a}_m + n \ddot{a}_x = \ddot{a}_m + n E_x \cdot \ddot{a}_{x+n}$$

Most important identity

$$1 = \delta \ddot{a}_x + \dddot{A}_x$$

$$\ddot{a}_x = \frac{1 - \dddot{A}_x}{\delta}$$

$$\dddot{A}_x:n = 1 - \delta \ddot{a}_{x:n}$$

$$\ddot{A}_{x:n} = 1 - (2\delta) \ddot{a}_{x:n}$$

$$\ddot{a}_x = \frac{1 - A_x}{d}$$

$$\ddot{a}_{x:n} = \frac{1 - A_{x:n}}{d}$$

$$1 = d \ddot{a}_{x:n} + A_{x:n}$$

Recursion relations

$$\ddot{a}_x = \ddot{a}_{x:1} + vp_x \ddot{a}_{x+1}$$

$$\ddot{a}_x = \ddot{a}_{x:m} + n \ddot{a}_x$$

$$\ddot{a}_x = 1 + vp_x \ddot{a}_{x+1}$$

$$\ddot{a}_x = 1 + v^2 p_x \ddot{a}_{x+1}$$

$$\ddot{a}_{x:m} = 1 + vp_x \ddot{a}_{x+1:n-m}$$

$$\ddot{a}_{x:m} = \ddot{a}_{x:n} - n E_x \cdot \ddot{a}_{x+n}$$

$$\ddot{a}_x = vp_x + vp_x \ddot{a}_{x+1}$$

$$\ddot{a}_{x:m} = vp_x + vp_x \ddot{a}_{x+1:n-m}$$

$$(I \ddot{a})_x = 1 + vp_x (I \ddot{a})_{x+1} + \ddot{a}_{x+1}$$

$$\ddot{a}_x + vp_x (I \ddot{a})_{x+1}$$

Whole life annuity due: $\bar{a}_x$

$$\bar{a}_x = E[\bar{a}_{x:1}] = \sum_{k=0}^\infty v^k \cdot k p_x$$

$$\text{Var}[\bar{a}_{x:1}] = \frac{2A_x - (A_x)^2}{d^2}$$

$n$-yr temporary annuity due: $\bar{a}_{x:n}$

$$\bar{a}_{x:n} = E[Y] = \sum_{k=0}^{n-1} v^k \cdot k p_x$$

$$\text{Var}[Y] = \frac{2A_{x:n} - (A_{x:n})^2}{d^2}$$

$n$-yr deferred annuity due: $\bar{a}_{x:n}$

$$\bar{a}_{x:n} = E[Y] = \sum_{k=n}^\infty v^k \cdot k p_x$$

$$\text{Var}[Y] = \frac{2A_{x:n} - (A_{x:n})^2}{d^2}$$

$n$-yr certain and life annuity due: $\bar{a}_{x:n}$

$$\bar{a}_{x:n} = \ddot{a}_x - \ddot{a}_{x:n}$$

$$\bar{a}_{x:n} = n E_x \cdot \ddot{a}_{x+n}$$

$n$-yr deferred annuity due: $\ddot{a}_{x:n}$

$$\ddot{a}_{x:n} = \ddot{a}_x + \ddot{a}_m - \ddot{a}_{x:n}$$

$$\ddot{a}_{x:n} = \ddot{a}_m + \sum_{k=0}^{n-1} v^k \cdot k p_x$$

$$\ddot{a}_{x:n} = \ddot{a}_m + n \ddot{a}_x$$
Whole life immediate: $a_x$

$$a_x = E[\bar{a}_x] = \sum_{k=1}^{\infty} v^k \cdot kp_x$$

$$\bar{a}_x = \frac{1 - (1 + i)A_x}{i}$$

$m$-thly annuities

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}}$$

$$Var[Y] = \frac{2A_x^{(m)} - (A_x^{(m)})^2}{(d^{(m)})^2}$$

$$\bar{a}_x^{(m)} = \bar{a}_x - \frac{1}{m}$$

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{1}{m}(1 - nE_x)$$

Accumulation function:

$$\bar{s}_{x:\overline{m}} = \frac{\bar{a}_{x:\overline{m}}}{nE_x} = \int_0^n \frac{1}{n-tE_{x+t}} dt$$

Limit of interest rate $i = 0$:

$$a_x \xrightarrow{i=0} e_x$$

$$\ddot{a}_x \xrightarrow{i=0} 1 + e_x$$

$$\bar{a}_x \xrightarrow{i=0} e_x$$

$$a_{x:\overline{m}} \xrightarrow{i=0} e_{x:\overline{m}}$$

$$\ddot{a}_{x:\overline{m}} \xrightarrow{i=0} 1 + e_{x:\overline{m}}$$

$$\bar{a}_{x:\overline{m}} \xrightarrow{i=0} e_{x:\overline{m}}$$
Chapter 6: Benefit Premiums

Loss function:
Loss = PV of Benefits – PV of Premiums

Fully continuous equivalence premiums
(whole life and endowment only):

\[ P(\bar{A}_x) = \frac{\tilde{A}_x}{\bar{a}_x} \]
\[ \tilde{P}(\bar{A}_x) = \frac{\delta \tilde{A}_x}{1 - \tilde{A}_x} \]
\[ \bar{P}(\bar{A}_x) = \frac{1}{\bar{a}_x - \delta} \]

\[ Var[L] = \left(1 + \frac{\tilde{P}}{\delta}\right)^2 \left[2\tilde{A}_x - (\tilde{A}_x)^2\right] \]
\[ Var[L] = \frac{2\tilde{A}_x - (\tilde{A}_x)^2}{(\delta \tilde{a}_x)^2} \]
\[ Var[L] = \frac{2\tilde{A}_x - (\tilde{A}_x)^2}{(1 - \tilde{A}_x)^2} \]

Fully discrete equivalence premiums
(whole life and endowment only):

\[ P(A_x) = \frac{A_x}{\bar{a}_x} = P_x \]
\[ P(A_x) = \frac{dA_x}{1 - A_x} \]
\[ P(A_x) = \frac{1}{\bar{a}_x - d} \]

\[ Var[L] = \left(1 + \frac{P}{d}\right)^2 \left[2A_x - (A_x)^2\right] \]
\[ Var[L] = \frac{2A_x - (A_x)^2}{(d \bar{a}_x)^2} \]
\[ Var[L] = \frac{2A_x - (A_x)^2}{(1 - A_x)^2} \]

Semicontinuous equivalence premiums:

\[ P(\tilde{A}_x) = \frac{\tilde{A}_x}{\bar{a}_x} \]

\[ m\text{-thly equivalence premiums:} \]

\[ P^{(m)} = \frac{A^{(m)}}{\bar{a}^{(m)}} \]

\[ h\text{-payment insurance premiums:} \]

\[ h\tilde{P}(\tilde{A}_x) = \frac{\tilde{A}_x}{\tilde{a}_x} \]
\[ h\bar{P}(\tilde{A}_x; n) = \frac{\bar{A}_x}{\bar{a}_x} \]
\[ hP_x = \frac{A_x}{\bar{a}_x} \]
\[ hP_x; n = \frac{A_x}{\bar{a}_x} \]

Pure endowment annual premium \( P_{x,\overline{1}} \):
It is the reciprocal of the actuarial accumulated value \( \overline{s}_{x,\overline{1}} \) because the share of the survivor who has deposited \( P_{x,\overline{1}} \) at the beginning of each year for \( n \) years is the contractual $1 pure endowment, i.e.

\[ P_{x,\overline{1}}\overline{s}_{x,\overline{1}} = 1 \] (1)

\( P \) minus \( P \) over \( P \) problems:
The difference in magnitude of level benefit premiums is solely attributable to the investment feature of the contract. Hence, comparisons of the policy values of survivors at age \( x + n \) may be done by analyzing future benefits:

\[ (nP_x - P_{x,\overline{1}})\overline{s}_{x,\overline{1}} = A_{x+n} = \frac{nP_x - P_{x,\overline{1}}}{P_{x,\overline{1}}} \]
\[ (P_{x,\overline{1}} - nP_x)\overline{s}_{x,\overline{1}} = 1 - A_{x+n} = \frac{P_{x,\overline{1}} - nP_x}{P_{x,\overline{1}}} \]
\[ (P_{x,\overline{1}} - P_{x,\overline{1}})\overline{s}_{x,\overline{1}} = 1 = \frac{P_{x,\overline{1}} - P_{x,\overline{1}}}{P_{x,\overline{1}}} \]

Miscellaneous identities:

\[ \tilde{A}_{x;\overline{1}} = \frac{\bar{P}(\tilde{A}_{x;\overline{1}})}{P(\tilde{A}_{x;\overline{1}}) + \delta} \]
\[ A_{x;\overline{1}} = \frac{P_{x;\overline{1}}}{P_{x;\overline{1}} + d} \]
\[ \bar{a}_{x;\overline{1}} = \frac{1}{P(A_{x;\overline{1}}) + \delta} \]
\[ \tilde{a}_{x;\overline{1}} = \frac{1}{P_{x;\overline{1}} + d} \]
Chapter 7: Benefit Reserves

Benefit reserve \(tV\):
The expected value of the prospective loss at time \(t\).

Continuous reserve formulas:
Prospective: \(tV(\bar{A}_x) = \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t}\)
Retrospective: \(tV(\bar{A}_x) = \bar{P}(\bar{A}_x)\bar{s}_{x;\bar{t}} - \frac{\bar{A}^1_{x;\bar{t}}}{\bar{t}E_x}\)
Premium diff.: \(tV(\bar{A}_x) = [\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)] \bar{a}_{x+t}\)
Paid-up Ins.: \(tV(\bar{A}_x) = \left[ 1 - \frac{\bar{P}(\bar{A}_x)}{\bar{P}(\bar{A}_{x+t})} \right] \bar{A}_{x+t}\)
Annuity res.: \(tV(\bar{A}_x) = 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}\)
Death ben.: \(tV(\bar{A}_x) = \frac{\bar{A}_{x+t} - \bar{A}_x}{1 - \bar{a}_x}\)
Premium res.: \(tV(\bar{A}_x) = \frac{\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)}{\bar{P}(\bar{A}_{x+t}) + \delta}\)

Discrete reserve formulas:
\(kV_x = A_{x+k} - P_k\bar{a}_{x+k}\)
\(kV_{x:n} = \left[ P_{x+k:;n-k} - P_{x:n-k} \right] \bar{a}_{x+k:;n-k}\)
\(kV_{x:n} = \left[ 1 - \frac{P_{x:n-k}}{P_{x+k:;n-k}} \right] A_{x+k:;n-k}\)
\(tV_{x:n} = P_{x:n:} \bar{s}_{x:n} - t\bar{k}_x\)
\(tV_x = 1 - \frac{\bar{a}_{x+k}}{\bar{a}_x}\)
\(kV_x = \frac{P_{x+k} - P_x}{P_{x+k} + \delta}\)
\(kV_{x:} = \frac{A_{x+k} - \bar{A}_x}{1 - \bar{a}_x}\)

\(h\)-payment reserves:
\(\bar{h}V_x = \bar{A}_{x+t:;n-k} - hP(\bar{A}_{x:;n-k})\bar{a}_{x+t:;h-k}\)
\(\bar{h}V_{x:n} = A_{x+k:;n-k} - hP_{x:n-k} \bar{a}_{x+k:;h-k}\)
\(\bar{h}V_{x:;n} = \bar{A}_{x+k:;n-k} - hP(\bar{A}_{x:;n-k})\bar{a}_{x+k:;h-k}\)
\(\bar{h}V_{x:;n} = A_{x+k:;n-k} - hP_{x:n-k} \bar{a}_{x+k:;h-k}\)

Variance of the loss function:
\(Var[\ell L] = \left( 1 + \frac{\bar{P}}{\delta} \right)^2 [2\bar{A}_{x+t} - (\bar{A}_{x+t})^2]\)
\(Var[\ell L] = \frac{2\bar{A}_{x+t} - (\bar{A}_{x+t})^2}{(1 - \bar{A}_x)^2} \) assuming EP
\(Var[\ell L] = \left( 1 + \frac{\bar{P}}{\delta} \right)^2 [2\bar{A}_{x+t:;n-k} - (\bar{A}_{x+t:;n-k})^2]\)
\(Var[\ell L] = \frac{2\bar{A}_{x+t:;n-k} - (\bar{A}_{x+t:;n-k})^2}{(1 - \bar{A}_{x:n-k})^2} \) assuming EP

Cost of insurance: funding of the accumulated costs of the death claims incurred between age \(x\) and \(x + t\) by the living at \(t\), e.g.
\(4k_x = \frac{d_x(1 + i)^3 + d_{x+1}(1 + i)^2 + d_{x+2}(1 + i) + d_{x+3}}{l_{x+4}}\)
\(= \frac{A_{x:;n}^1}{4E_x}\)
\(t_k_x = \frac{d_x}{l_{x+1}} = \frac{q_x}{p_x}\)

Accumulated differences of premiums:
\((nP_x - P_{x:;n})\bar{s}_{x:;n} = nV_x - nV_{x:;n}\)
\(= A_{x+n} - 0 = A_{x+n}\)
\((nP_x - P_x)\bar{s}_{x:;n} = nV_x - nV_x\)
\(= P_x\bar{a}_{x+n}\)
\((P_{x:;n} - P_x)\bar{s}_{x:;n} = nV_{x:;n} - nV_x\)
\(= 1 - nV_x\)
\((mP_{x:;m} - mP_x)\bar{s}_{x:;m} = mV_{x:;m} - mV_x\)
\(= A_{x+m:;n-m} - A_{x+m}\)

Relation between various terminal reserves (whole life/endorsement only):
\(m+n+pV_x = 1 - (1 - mV_x)(1 - nV_{x+m})(1 - pV_{x+m+n})\)
Chapter 8: Benefit Reserves

Notations:
- \( b_j \): death benefit payable at the end of year of death for the \( j \)-th policy year
- \( \pi_{j-1} \): benefit premium paid at the beginning of the \( j \)-th policy year
- \( b_t \): death benefit payable at the moment of death
- \( \pi_t \): annual rate of benefit premiums payable continuously at \( t \)

Benefit reserve:
\[
hV = \sum_{j=0}^{\infty} b_{h+j+1} v^{j+1} j p_{x+h} q_{x+h+j} - \sum_{j=0}^{\infty} \pi_{h+j} v^{j} j p_{x+h}
\]
\[
i\hat{V} = \int_0^\infty b_{t+u} v^u u p_{x+t} \mu_x(t+u) du - \sum_{u=0}^{\infty} \pi_{t+u} v^u u p_{x+t} du
\]

Recursion relations:
\[
hV + \pi_h = v q_{x+h} \cdot b_{h+1} + v p_{x+h} \cdot h+1V
\]
\[
(hV + \pi_h)(1 + i) = q_{x+h} \cdot b_{h+1} + p_{x+h} \cdot h+1V
\]
\[
(hV + \pi_h)(1 + i) = h+1V + q_{x+h}(b_{h+1} - h+1V)
\]

Terminology:
- “policy year \( h+1 \)” \( \equiv \) the policy year from time \( t = h \) to time \( t = h + 1 \)
- “\( hV + \pi_h \)” \( \equiv \) initial benefit reserve for policy year \( h+1 \)
- “\( hV \)” \( \equiv \) terminal benefit reserve for policy year \( h \)
- “\( h+1V \)” \( \equiv \) terminal benefit reserve for policy year \( h + 1 \)

Net amount at Risk for policy year \( h+1 \)
\[
\text{Net Amount Risk} \equiv b_{h+1} - h+1V
\]

When the death benefit is defined as a function of the reserve:
For each premium \( P \), the cost of providing the ensuing year’s death benefit, based on the net amount at risk at age \( x + h \), is: \( v q_{x+h}(b_{h+1} - h+1V) \). The leftover, \( P - v q_{x+h}(b_{h+1} - h+1V) \) is the source of reserve creation. Accumulated to age \( x + n \), we have:
\[
nV = \sum_{h=0}^{n-1} [P - v q_{x+h}(b_{h+1} - h+1V)] (1 + i)^{n-h}
\]
\[
= Ps_{\bar{m}} - \sum_{h=0}^{n-1} v q_{x+h}(b_{h+1} - h+1V)(1 + i)^{n-h}
\]

- If the death benefit is equal to the benefit reserve for the first \( n \) policy years
  \( nV = Ps_{\bar{m}} \)
- If the death benefit is equal to $1 plus the benefit reserve for the first \( n \) policy years
  \( nV = Ps_{\bar{m}} - \sum_{h=0}^{n-1} v q_{x+h}(1 + i)^{n-h} \)
• If the death benefit is equal to $1 plus the benefit reserve for the first \( n \) policy years and \( q_{x+h} \equiv q \) constant

\[
nV = P \bar{s}_n - vq \bar{s}_n = (P - vq) \bar{s}_n
\]

Reserves at fractional durations:

\[
(hV + \pi_h)(1 + i)^s = sp_{x+h} \cdot v^{1-s} + s q_{x+h} v^{1-s} \]

\[
\text{UDD} \Rightarrow (hV + \pi_h)(1 + i)^s = (1 - s \cdot q_{x+h}) h_s V + s \cdot q_{x+h} v^{1-s} = h_s V + s \cdot q_{x+h} (v^{1-s} - h_s V)
\]

\[
h_{s+1} V = v^{1-s} \cdot s q_{x+h} \cdot b_{x+1} + v^{1-s} \cdot s p_{x+h} \cdot h_{+1} V
\]

\[
\text{UDD} \Rightarrow h_{s+1} V = (1 - s)(hV + \pi_h) + s(h+1V)
\]

\[
i.e. \quad h_{s+1} V = (1 - s)(hV) + s(h+1V) + (1 - s)(\pi_h)
\]

\[\text{unearned premium}\]

Next year losses:

\[
\Lambda_h \equiv \text{losses incurred from time } h \text{ to } h + 1
\]

\[
E[\Lambda_h] = 0
\]

\[
Var[\Lambda_h] = v^2(b_{h+1} - h_{+1}V)^2 p_{x+h} q_{x+h}
\]

The Hattendorf theorem

\[
Var[hL] = Var[\Lambda_h] + v^2 p_{x+h} Var[h+1L]
\]

\[
= v^2(b_{h+1} - h_{+1}V)^2 p_{x+h} \cdot q_{x+h} + v^2 p_{x+h} Var[h+1L]
\]

\[
Var[hL] = v^2(b_{h+1} - h_{+1}V)^2 p_{x+h} \cdot q_{x+h}
\]

\[
+ v^4(b_{h+2} - h_{+2}V)^2 p_{x+h} \cdot p_{x+h+1} \cdot q_{x+h+1}
\]

\[
+ v^6(b_{h+3} - h_{+3}V)^2 p_{x+h} \cdot p_{x+h+1} \cdot p_{x+h+2} \cdot q_{x+h+2} + \cdots
\]
Chapter 9: Multiple Life Functions

Joint survival function:

\[ s_{T(x)T(y)}(s, t) = Pr[T(x) > s \& T(y) > t] \]

\[ t_p_{xy} = s_{T(x)T(y)}(t, t) \]

\[ = Pr[T(x) > t \text{ and } T(y) > t] \]

Joint life status:

\[ F_T(t) = Pr[\min(T(x), T(y)) \leq t] \]

\[ = t_{q_{xy}} \]

\[ = 1 - t_p_{xy} \]

Independant lives

\[ t_{p_{xy}} = t_{p_x} \cdot t_{p_y} \]

\[ t_{q_{xy}} = t_{q_x} + t_{q_y} - t_{q_x} \cdot t_{q_y} \]

Complete expectation of the joint-life status:

\[ \hat{e}_{xy} = \int_0^\infty t_p_{xy} dt \]

PDF joint-life status:

\[ f_{T(xy)}(t) = t_p_{xy} \cdot \mu_{xy}(t) \]

\[ \mu_{xy}(t) = \frac{f_{T(xy)}(t)}{1 - F_{T(xy)}(t)} = \frac{f_{T(xy)}(t)}{t_p_{xy}} \]

Independant lives

\[ \mu_{xy}(t) = \mu(x + t) + \mu(y + t) \]

\[ f_{T(xy)}(t) = t_{p_x} \cdot t_{p_y} \mu(x + t) + \mu(y + t) \]

Curate joint-life functions:

\[ k_{p_{xy}} = k_{p_x} \cdot k_{p_y} \text{ [IL]} \]

\[ k_{q_{xy}} = k_{q_x} + k_{q_y} - k_{q_x} \cdot k_{q_y} \text{ [IL]} \]

\[ Pr[K = k] = k_{p_{xy}} - k + 1 \cdot p_{xy} \]

\[ = k_{p_{xy}} \cdot q_x + k_{q_y} + k \]

\[ = k_{p_{xy}} \cdot q_x + k_{q_y} + k \]

\[ q_x + t_{y-t} = q_x + k_{q_y} + k_{q_y} + k \text{ [IL]} \]

\[ e_{xy} = E[K(xy)] = \sum_1^\infty k_{p_{xy}} \]

Last survivor status \( T(xy) \):

\[ T(xy) + T(xy) = T(x) + T(y) \]

\[ T(xy) \cdot T(xy) = T(x) \cdot T(y) \]

\[ f_T(xy) + f_T(xy) = f_T(x) + f_T(y) \]

\[ F_T(xy) + F_T(xy) = F_T(x) + F_T(y) \]

\[ t_p_{xy} + t_p_{xy} = t_p_x + t_p_y \]

\[ A_x + A_y \]

\[ a_x + a_y \]

\[ \hat{e}_x + \hat{e}_y \]

\[ e_x + e_y \]

\[ n_1 q_{xy} = n_1 q_x + n_1 q_y - n_1 q_{xy} \]

Complete expectation of the last-survivor status:

\[ \hat{e}_{xy} = \int_0^\infty t_{p_{xy}} dt \]

\[ e_{xy} = \sum_1^n k_{p_{xy}} \]

Variances:

\[ Var[T(u)] = 2 \int_0^\infty t \cdot t_p_u dt - (\hat{e}_u)^2 \]

\[ Var[T(xy)] = 2 \int_0^\infty t \cdot t_p_{xy} dt - (\hat{e}_{xy})^2 \]

\[ Var[T(xy)] = 2 \int_0^\infty t \cdot t_p_{xy} dt - (\hat{e}_{xy})^2 \]

Notes:

For joint-life status, work with \( p \)'s:

\[ n_{p_{xy}} = n_{p_x} \cdot n_{p_y} \]

For last-survivor status, work with \( q \)'s:

\[ n_1 q_{xy} = n_1 q_x \cdot n_1 q_y \]

“Exactly one” status:

\[ n_{p_{xy}}^{[1]} = n_{p_{xy}} - n_{p_{xy}} \]

\[ = n_{p_x} + n_{p_y} - 2 n_{p_x} \cdot n_{p_y} \]

\[ = n_{q_x} + n_{q_y} - 2 n_{q_x} \cdot n_{q_y} \]

\[ \hat{a}_{xy}^{[1]} = \hat{a}_x + \hat{a}_y - 2 \hat{a}_{xy} \]
Common shock model:

\[ s_{T(x)}(t) = s_{T^*(x)}(t) \cdot s_z(t) \]
\[ = s_{T^*(x)}(t) \cdot e^{-\lambda t} \]
\[ s_{T(y)}(t) = s_{T^*(y)}(t) \cdot s_z(t) \]
\[ = s_{T^*(y)}(t) \cdot e^{-\lambda t} \]
\[ s_{T(x)T(y)}(t) = s_{T^*(x)}(t) \cdot s_{T^*(y)}(t) \cdot s_z(t) \]
\[ = s_{T^*(x)}(t) \cdot s_{T^*(y)}(t) \cdot e^{-\lambda t} \]
\[ \mu_{xy}(t) = \mu(x + t) + \mu(y + t) + \lambda \]

Insurance functions:

\[ A_u = \sum_{k=0}^{\infty} v^{k+1} \cdot k^p_u \cdot q_{u+k} \]
\[ = \sum_{k=0}^{\infty} v^{k+1} P_t[K = k] \]
\[ A_{xy} = \sum_{k=0}^{\infty} v^{k+1} \cdot k^p_{xy} \cdot q_{x+k,y+k} \]
\[ A_{xy} = \sum_{k=0}^{\infty} v^{k+1} \cdot k^p_{xy} \cdot q_{x+k,y+k} \]
\[ Var[Z] = 2A_u - (A_u)^2 \]
\[ Var[Z] = 2A_{xy} - (A_{xy})^2 \]
\[ Cov[v^{(xy)}, v^{(xy)}] = (A_x - \bar{A}_x)(A_y - \bar{A}_y) \]

Variance of insurance functions:

\[ Cov[T(xy), T(xy)] = Cov[T(x), T(y)] + \{E[T(x)] - E[T(xy)]\} \cdot \{(E[T(y)] - E[T(xy)]\} \]
\[ = Cov[T(x), T(y)] + (\hat{c}_x - \hat{c}_{xy})(\hat{c}_y - \hat{c}_{xy}) \]
\[ = (\hat{c}_x - \hat{c}_{xy})(\hat{c}_y - \hat{c}_{xy}) \] [IL]

Insurances:

\[ \bar{A}_x = 1 - \delta\bar{a}_x \]
\[ \bar{A}_{xy} = 1 - \delta\bar{a}_{xy} \]
\[ \bar{A}_{xy} = 1 - \delta\bar{a}_{xy} \]

Premiums:

\[ P_x = \frac{1}{\bar{a}_x} - d \]
\[ P_{xy} = \frac{1}{\bar{a}_{xy}} - d \]
\[ P_{xy} = \frac{1}{\bar{a}_{xy}} - d \]

Annuity functions:

\[ \bar{a}_u = \int_0^{\infty} v^t \cdot t^p_u dt \]
\[ var[Y] = \frac{2\bar{A}_u - (\bar{A}_u)^2}{\bar{a}_u^2} \]

Reversionary annuities:

A reversionary annuity is payable during the existence of one status u only if another status v has failed. E.g. an annuity of 1 per year payable continuously to (y) after the death of (x).

\[ \bar{a}_{xy} = \bar{a}_y - \bar{a}_{xy} \]
Chapter 10 & 11: Multiple Decrement Models

Notations:

\[ t q_x^{(j)} = \text{probability of decrement in the next } t \text{ years due to cause } j \]

\[ t q_x^{(\tau)} = \text{probability of decrement in the next } t \text{ years due to all causes} = \sum_{j=1}^{m} t q_x^{(j)} \]

\[ \mu_x^{(j)} = \text{the force of decrement due only to decrement } j \]

\[ \mu_x^{(\tau)} = \text{the force of decrement due to all causes simultaneously} = \sum_{j=1}^{m} \mu_x^{(j)} \]

\[ t p_x^{(\tau)} = \text{probability of surviving } t \text{ years despite all decrements} = 1 - t q_x^{(\tau)} = e^{-\int_0^t \mu_x^{(\tau)}(s)ds} \]

Derivative:

\[ \frac{d}{dt} (t p_x^{(\tau)}) = -\frac{d}{dt} (t q_x^{(\tau)}) = - t p_x^{(\tau)} \mu_x^{(\tau)} \]

Integral forms of \( t q_x \):

\[ t q_x^{(j)} = \int_0^t s p_x^{(\tau)} \cdot \mu_x^{(j)}(s)ds \]

\[ t q_x^{(\tau)} = \int_0^t s p_x^{(\tau)} \cdot \mu_x^{(\tau)}(s)ds \]

Probability density functions:

Joint PDF:

\[ f_{T,J}(t, j) = t p_x^{(\tau)} \cdot \mu_x^{(j)}(t) \]

Marginal PDF of \( J \):

\[ f_J(j) = \int_0^\infty f_{T,J}(t, j)dt \]

Marginal PDF of \( T \):

\[ f_T(t) = t p_x^{(\tau)} \cdot \mu_x^{(\tau)}(t) = \sum_{j=1}^{m} f_{T,J}(t, j) \]

Conditional PDF:

\[ f_{J|T}(j|t) = \frac{\mu_x^{(j)}(t)}{\mu_x^{(\tau)}(t)} \]

Survivorship group:

Group of \( l^{(\tau)}_a \) people at some age \( a \) at time \( t = 0 \). Each member of the group has a joint pdf for time until decrement and cause of decrement.

\[ n d_x^{(j)} = \frac{l^{(\tau)}_a \cdot x-a p_a^{(\tau)} \cdot n q_x^{(j)}}{x-a+n} = l^{(\tau)}_a \int_{x-a} t p_x^{(\tau)} \cdot \mu_x^{(j)}(t)dt \]

\[ n d_x^{(\tau)} = \sum_{j=1}^{m} n d_x^{(j)} \]

\[ l^{(\tau)}_a = \sum_{j=1}^{m} l^{(j)}_a \]

\[ q_x^{(j)} = \frac{d_x^{(j)}}{l^{(\tau)}_a} \]

Associated single decrement:

\[ t q_x^{(j)} = \text{probability of decrement from cause } j \text{ only} \]

\[ t p_x^{(j)} = \exp \left[-\int_0^t \mu_x^{(j)}(s)ds \right] = 1 - t q_x^{(j)} \]
Basic relationships:

\[ t_p(x) = \exp \left\{ -\int_0^t \left[ \mu_x(t) + \cdots + \mu_x(m)(t) \right] dt \right\} \]

\[ t_p(x) = \prod_{i=1}^m t_{p_x}^{(i)} \]

\[ t_q(x) \geq t_q(x) \]

\[ t_p(x) \geq t_p(x) \]

UDD for multiple decrements:

\[ t_q(x) = t \cdot q_x(t) \]

\[ t_q(x) = t \cdot q_x(t) \]

\[ q_x(t) = t_p(x) \cdot \mu_x(t) \]

\[ \mu_x(t) = \frac{q_x(t)}{t_p(t)} = \frac{q_x(t)}{1-t \cdot q_x(t)} \]

Decrement uniformly distributed in the associated single decrement table:

\[ t_q(x) = t \cdot q_x(t) \]

\[ q_x^{(1)} = q_x^{(1)} \left( 1 - \frac{1}{2} q_x^{(2)} \right) \]

\[ q_x^{(2)} = q_x^{(2)} \left( 1 - \frac{1}{2} q_x^{(1)} \right) \]

\[ q_x^{(1)} = q_x^{(1)} \left( 1 - \frac{1}{2} q_x^{(2)} - \frac{1}{2} q_x^{(3)} + \frac{1}{3} q_x^{(2)} \cdot q_x^{(3)} \right) \]

Actuarial present values

\[ A = \sum_{j=1}^m \left[ \int_0^\infty B_j(x) \cdot t_p(x) \cdot \mu_x(t) dt \right] \]

Instead of summing the benefits for each possible cause of death, it is often easier to write the benefit as one benefit given regardless of the cause of death and add/subtract other benefits according to the cause of death.

Premiums:

\[ P_x^{(\tau)} = \sum_{k=0}^\infty B_{k+1}^{(\tau)} \cdot (k+1) \cdot k_p(x) \cdot q_x^{(\tau)} \]

\[ P_x^{(j)} = \sum_{k=0}^\infty B_{k+1}^{(j)} \cdot (k+1) \cdot k_p(x) \cdot q_x^{(j)} \]
Chapter 15: Models Including Expenses

Notations:

\[ G \equiv \text{expense loaded (or gross) premium} \]
\[ b \equiv \text{face amount of the policy} \]
\[ G/b \equiv \text{per unit gross premium} \]

Expense policy fee:
The portion of \( G \) that is independent of \( b \).

Asset shares notations:

\[ G \equiv \text{level annual contract premium} \]
\[ k\text{AS} \equiv \text{asset share assigned to the policy at time } t = k \]
\[ c_k \equiv \text{fraction of premium paid for expenses at } k \text{ (i.e. } cG \text{ is the expenses premium)} \]
\[ e_k \equiv \text{expenses paid per policy at time } t = k \]
\[ q^{(d)}_{x+k} \equiv \text{probability of decrement by death} \]
\[ q^{(w)}_{x+k} \equiv \text{probability of decrement by withdrawal} \]
\[ kCV \equiv \text{cash amount due to the policy holder as a withdrawal benefit} \]
\[ b_k \equiv \text{death benefit due at time } t = k \]

Recursion formula:

\[
[k\text{AS} + G(1 - c_k) - e_k](1 + i) = q^{(d)}_{x+k} \cdot b_{k+1} + q^{(w)}_{x+k} \cdot kCV + p^{(\tau)}_{x+k} \cdot k+1\text{AS} \\
= k+1\text{AS} + q^{(d)}_{x+k}(b_{k+1} - k+1\text{AS}) + q^{(w)}_{x+k}(kCV - k+1\text{AS})
\]

Direct formula:

\[
n\text{AS} = \sum_{h=0}^{n-1} \frac{G(1 - c_h) - e_h - vq^{(d)}_{x+h}b_{h+1} - vq^{(w)}_{x+h}h+1CV}{n-hE^{(\tau)}_{x+h}}
\]
Chapter 3

\[ \mu(x) = \mu > 0, \forall x \]
\[ s(x) = e^{-\mu x} \]
\[ l_x = l_0 e^{-\mu x} \]
\[ n p_x = e^{-n \mu} (p_x)^n \]
\[ \dot{e}_x = \frac{1}{\mu} = E[T] = E[X] \]
\[ \ddot{e}_{x:\overline{n}} = \dot{e}_x (1 - np_x) \]
\[ Var[T] = Var[x] = \frac{1}{\mu^2} \]
\[ m_x = \mu \]
\[ Median[T] = \frac{\ln 2}{\mu} = Median[X] \]
\[ e_x = \frac{p_x}{q_x} = E[K] \]
\[ Var[K] = \frac{p_x}{(q_x)^2} \]

Chapter 4

\[ A_x = \frac{\mu}{\mu + \delta} \]
\[ \ddot{A}_x = \frac{\mu}{\mu + 2\delta} \]
\[ \overline{A}_{x:n} = A_x (1 - n E_x) \]
\[ n E_x = e^{-n(\mu+\delta)} \]
\[ (TA)_x = \frac{\mu}{(\mu + \delta)^2} \]
\[ A_x = \frac{q}{q + i} \]
\[ \ddot{A}_x = \frac{q}{q + 2i + i^2} \]
\[ A_{x:\overline{n}} = A_x (1 - n E_x) \]

Chapter 5

\[ a_x = \frac{1}{\mu + \delta} \]
\[ \ddot{a}_x = \frac{1}{\mu + 2\delta} \]
\[ \dot{a}_x = \frac{1 + i}{q + i} \]
\[ \ddot{a}_x = \frac{(1 + i)^2}{q + 2i + i^2} \]
\[ a_{x:\overline{n}} = a_x (1 - n E_x) \]
\[ \ddot{a}_{x:\overline{n}} = \ddot{a}_x (1 - n E_x) \]
\[ (TA)_x = A_x \dot{a}_x = \frac{\mu}{(\mu + \delta)^2} \]
\[ (IA)_x = \dot{A}_x \ddot{a}_x = \frac{\mu(1 + i)}{(\mu + \delta)(q + i)} \]
\[ (IA)_x = A_x \ddot{a}_x = \frac{q(1 + i)}{(q + i)^2} \]
\[ (TA)_x = (\ddot{a}_x)^2 = \frac{1}{(\mu + \delta)^2} \]
\[ (I\ddot{a})_x = (\ddot{a}_x)^2 = \left(\frac{1 + i}{q + i}\right)^2 \]

Chapter 6

\[ P_x = u q_x = P_x^{\overline{n}} \]
\[ \overline{P}(A_x) = \mu = \overline{P}(A_{x:\overline{n}}) \]

For fully discrete whole life, w/ EP,
\[ Var[Loss] = p \cdot \ddot{A}_x \]

For fully continuous whole life, w/EP,
\[ Var[Loss] = \overline{\ddot{A}}_x \]

Chapter 7

\[ i \overline{V}(A_x) = 0, \ t \geq 0 \]
\[ k \overline{V}_x = 0, \ k = 0,1,2, \ldots \]

For fully discrete whole life, assuming EP,
\[ Var[k Loss] = p \cdot \ddot{A}_x, \ k = 0,1,2, \ldots \]

For fully continuous whole life, assuming EP,
\[ Var[i Loss] = \overline{\ddot{A}}_x, \ t \geq 0 \]

Chapter 9

For two constant forces, i.e. \( \mu^M \) acting on \((x)\) and \( \mu^F \) acting on \((y)\), we have:
\[ \overline{A}_{xy} = \frac{\mu^M + \mu^F}{\mu^M + \mu^F + \delta} \]
\[ \overline{a}_{xy} = \frac{1}{\mu^M + \mu^F + \delta} \]
\[ \dot{e}_{xy} = \frac{1}{\mu^M + \mu^F} \]
\[ A_{xy} = \frac{q_{xy}}{q_{xy} + i} \]
\[ \dot{a}_{xy} = \frac{1 + i}{q_{xy} + i} \]
\[ e_{xy} = \frac{p_{xy}}{q_{xy}} \]
De Moivre’s Law

• Chapter 3

\[ s(x) = 1 - \frac{x}{\omega} \]

\[ l_x = l_0 \left( \frac{\omega - x}{\omega} \right)^c \propto (\omega - x)^c \]

\[ q_x = \mu(x) = \frac{1}{\omega - x} \]

\[ n|n|q_x = \frac{m}{\omega - x} = \frac{\omega - x - n}{\omega - x} \]

\[ n p_x = \frac{\omega - x}{\omega - x} \]

\[ t p_x \mu(x + t) = q_x = \mu(x) = f_T(x), \ 0 \leq t < \omega - x \]

\[ L_x = l_x + l_{x+1} \]

\[ \hat{e}_x = \frac{\omega - x}{2} = E[T] = \text{Median}[T] \]

\[ e_x = \frac{\omega - x}{2} - \frac{1}{2} = E[K] \]

\[ \text{Var}[T] = \frac{(\omega - x)^2}{12} \]

\[ \text{Var}[K] = \frac{(\omega - x)^2 - 1}{12} \]

\[ m_x = \frac{q_x}{1 - 2q_x} = \frac{2d_x}{l_x + l_{x+1}} \]

\[ a(x) = E[S] = \frac{1}{2} \]

\[ \hat{e}_{x:n} = \hat{e}_{x-n} = n \ n p_x + \frac{n}{2} \ n q_x \]

\[ \hat{e}_{x:n} = e_{x:n} + \frac{n}{2} \ n q_x \]

• Chapter 4

\[ \bar{A}_x = \frac{\bar{a}_{\omega-x}}{\omega - x} \]

\[ \bar{A}^{1}_{x:n} = \frac{\bar{a}_{\omega-x}}{\omega - x} \]

\[ 2 \bar{A}_x = \frac{\bar{a}_{2(\omega-x)}}{2(\omega - x)} \]

\[ A_x = \frac{\bar{a}_{\omega-x}}{\omega - x} \]

\[ A^{1}_{x:n} = \frac{\bar{a}_{\omega-x}}{\omega - x} \]

\[ (TA)_x = \frac{(Ta)_{\omega-x}}{\omega - x} \]

\[ (IA)_x = \frac{(Ia)_{\omega-x}}{\omega - x} \]

\[ (TA)^{1}_{x:n} = \frac{(Ta)_{\omega-x}}{\omega - x} \]

\[ (IA)^{1}_{x:n} = \frac{(Ia)_{\omega-x}}{\omega - x} \]

• Chapter 5

No useful formulas: use \( \bar{a}_x = \frac{1}{\omega} \bar{A}_x \) and the chapter 4 formulas.

• Chapter 9

\[ \hat{e}_{xx} = \frac{\omega - x}{3} \ (\equiv \text{MDML with } \mu = 2/(\omega - x)) \]

\[ \hat{e}_{x:j} = \frac{2(\omega - x)}{3} \]

\[ \hat{e}_{xy} = \frac{y-x p_x \hat{e}_{y} + y-x q_x \hat{e}_{y}}{\omega - x} \]

For two lives with different \( \omega \)'s, simply translate one of the age by the difference in \( \omega \)'s. E.g.

Age 30, \( \omega = 100 \) ⇔ Age 15, \( \omega = 85 \)

Modified De Moivre’s Law

• Chapter 3

\[ s(x) = \left(1 - \frac{x}{\omega} \right)^c \]

\[ l_x = l_0 \left( \frac{\omega - x}{\omega} \right)^c \propto (\omega - x)^c \]

\[ \mu(x) = \frac{c}{\omega - x} \]

\[ \mu(x) = \frac{c}{\omega - x} \]

\[ n p_x = \left( \frac{\omega - x - n}{\omega - x} \right)^c \]

\[ \hat{e}_x = \frac{\omega - x}{c + 1} = E[T] \]

\[ V a r[T] = \frac{(\omega - x)^2 c}{(c + 1)^2(c + 2)} \]

• Chapter 9

\[ \hat{e}_{xx} = \frac{\omega - x}{2c + 1} \]

\[ \equiv \hat{e}_x \text{ with } \mu = \frac{2c}{\omega - x} \]
Uniform Distribution of Deaths (UDD)

- Chapter 3
  \[ t q_x = t \cdot q_x, \quad 0 \leq t \leq 1 \]
  \[ \mu(x + t) = \frac{q_x}{1 - t q_x}, \quad 0 \leq t \leq 1 \]
  \[ l_{x+t} = l_x - t \cdot a_x, \quad 0 \leq t \leq 1 \]
  \[ s q_{x+t} = \frac{s q_x}{1 - t q_x}, \quad 0 \leq s + t \leq 1 \]
  \[ \text{Var}[T] = \text{Var}[K] + \frac{1}{12} \]
  \[ m_x = \mu(x + \frac{1}{2}) = - \frac{q_x}{1 - \frac{1}{2} q_x} \]
  \[ L_x = l_x \frac{d_x}{\mu(x)} \]
  \[ a(x) = \frac{1}{2} \]
  \[ \dot{\bar{A}}_{x:t} = P_x + \frac{1}{2} q_x \]
  \[ t p_x \mu(x + t) = q_x, \quad 0 \leq t \leq 1 \]

- Chapter 4
  \[ \bar{A}_x = \frac{i}{\delta} A_x \]
  \[ A_x^{(m)} = \frac{i}{i(m)} A_x \]
  \[ \bar{A}_x^{(m)} = \frac{i}{\delta} \bar{A}_x^{(m)} \]
  \[ (IA)_x^{(m)} = \frac{i}{\delta} (IA)_x^{(m)} \]
  \[ \bar{A}_x, \bar{A}_x^{(m)} = \frac{i}{\delta} A_x^{(m)} + A_x^{(m)} + \frac{i}{\delta} A_x^{(m)} + n E_x \]
  \[ n \bar{A}_x = \frac{i}{\delta} n A_x \]
  \[ 2 \bar{A}_x = 2 \frac{i}{\delta} + 2 \frac{i}{\delta} A_x \]

- Chapter 5
  \[ \bar{a}_x^{(m)} = \bar{a}_x - \frac{m - 1}{2 m} \]
  \[ \bar{a}_x^{(m)} = \alpha(m) \bar{a}_x - \beta(m) \]
  \[ \bar{a}_x^{(m)} = \alpha(m) \bar{a}_x^{(m)} - \beta(m) (1 - n E_x) \]
  \[ \text{with: } \alpha(m) = \frac{id}{i(m) \bar{d}(m)}, \quad 1 \]
  \[ \text{and } \beta(m) = \frac{i - i(m)}{i(m) \bar{d}(m)}, \quad m - 1 \]
  \[ \bar{a}_x^{(m)} = \bar{a}_x^{(m)} - n E_x \bar{a}_x^{(m)} \]

- Chapter 6
  \[ P(\bar{A}_x) = \frac{i}{\delta} P_x \]
  \[ P(\bar{A}_x^{(m)}) = \frac{i}{\delta} P_x^{(m)} \]
  \[ P(\bar{A}_x^{(m)}) = \frac{i}{\delta} P_x^{(m)} + P_x^{(m)} \]
  \[ P_x^{(m)} = \frac{1}{\delta} P_x^{(m)} + P_x^{(m)} \]
  \[ P_x^{(m)} = \frac{1}{\delta} P_x^{(m)} + P_x^{(m)} \]
  \[ P_x^{(m)} = \frac{1}{\delta} P_x^{(m)} + P_x^{(m)} \]

- Chapter 7
  \[ \bar{h} V^{(m)}(\bar{A}_x^{(m)}) = \bar{h} V(\bar{A}_x^{(m)}) + \beta(m) h P^{(m)}(\bar{A}_x^{(m)}) k V^{(1)}(\bar{A}_x^{(m)}) \]

- Chapter 10
  UDDMDT
  \[ q_x^{(j)} = t \cdot q_x^{(j)} \]
  \[ q_x^{(j)} = \mu_x^{(j)}(0) \]
  \[ q_x^{(j)} = \mu_x^{(j)}(0) \]
  \[ t u_x^{(j)} = \left( q_x^{(j)} \right)^{\frac{1}{\delta}} \]
  \[ \mu_x^{(j)} = \frac{q_x^{(j)}}{1 - t \cdot q_x^{(j)}} \]

  UDDASDT
  \[ q_x^{(j)} = t \cdot q_x^{(j)} \]
  \[ q_x^{(1)} = q_x^{(1)} \left( 1 - \frac{1}{2} q_x^{(2)} \right) \]
  \[ q_x^{(2)} = q_x^{(2)} \left( 1 - \frac{1}{2} q_x^{(1)} \right) \]
  \[ q_x^{(1)} = q_x^{(1)} \left( 1 - \frac{1}{2} q_x^{(2)} - \frac{1}{2} q_x^{(3)} + \frac{1}{2} q_x^{(2)} \cdot q_x^{(3)} \right) \]
Chapter 1: The Poisson Process

Poisson process with rate $\lambda$:

$$Pr[N(s + t) - N(s) = k] = \frac{e^{-\lambda t} (\lambda t)^k}{k!}$$

$$E[N(t)] = \lambda t$$

$$Var[N(t)] = \lambda t$$

Interarrival time distribution:

The waiting time between events. Let $T_n$ denote the time since occurrence of the event $n$. Then the $T_n$ are independent random variables following an exponential distribution with mean $1/\lambda$.

$$Pr[T_i \leq t] = 1 - e^{-\lambda t}$$

$$f_T(t) = \lambda e^{-\lambda t}$$

$$E[T] = \frac{1}{\lambda}$$

$$Var[T] = \frac{1}{\lambda^2}$$

Waiting time distribution:

Let $S_n$ be the time of the $n$-occurrence of the event, i.e. $S_n = \sum_{i=1}^{n} T_i$.

$S_n$ has a gamma distribution with parameters $n$ and $\theta = 1/\lambda$.

$$S_n \equiv \text{GammaRV}[\alpha = n, \theta = \frac{1}{\lambda}]$$

$$Pr[S_n \leq t] = \sum_{j=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^j}{j!}$$

$$f_{S_n}(t) = \frac{\lambda e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!}$$

$$E[S_n] = \frac{n}{\lambda}$$

$$Var[S_n] = \frac{n}{\lambda^2}$$

Two competing Poisson processes:

Probability that $n$ events in the Poisson process $(N_1, \lambda_1)$ occur before $m$ events in the Poisson process $(N_2, \lambda_2)$:

$$Pr[S_n^1 < S_m^2] = \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^k \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{n+m-1-k}$$

$$Pr[S_n^1 < S_1^2] = \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^n$$

$$Pr[S_1^1 < S_1^2] = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Sum of Poisson processes:

If $N_1, \cdots, N_k$ are independent Poisson processes with rates $\lambda_1, \cdots, \lambda_k$ then, $N = N_1 + \cdots + N_k$ is a Poisson process with rate $\lambda = \lambda_1 + \cdots + \lambda_k$.

Special events in a Poisson process:

Let $N$ be a Poisson process with rate $\lambda$. Some events $i$ are special with a probability $Pr[\text{event is special}] = \pi_i$ and $\tilde{N}_i$ counts the special events of kind $i$. Then, $\tilde{N}_i$ is a Poisson process with rate $\tilde{\lambda}_i = \pi_i \lambda$ and the $\tilde{N}_i$ are independent of one another.

If the probability $\pi_i(t)$ changes with time, then

$$E[\tilde{N}_i(t)] = \lambda \int_0^t \pi(s)ds$$

Non-homogeneous Poisson process:

$$\lambda(t) \equiv \text{intensity function}$$

$$m(t) \equiv \text{mean value function}$$

$$m(t) = \int_0^t \lambda(y)dy$$

$$Pr[N(t) = k] = e^{-m(t)} \left( \frac{m(t)}{k} \right)^k$$

Compound Poisson process:

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

$N(t)$ PoissonRV w/ rate $\lambda$

$$E[X(t)] = \lambda t \cdot E[Y]$$

$$Var[X(t)] = \lambda t \cdot Var[Y]$$
Chapter 2&3: Random Variables

\( k \)-th raw moment:
\[ \mu'_k = E[X^k] \]

\( k \)-th central moment:
\[ \mu_k = E[(X - \mu)^k] \]

Variance:
\[ \text{Var}[x] = \sigma^2 = E[X^2] - E[X]^2 \]
\[ = \mu_2 - \mu^2 \]

Standard deviation:
\[ \sigma = \sqrt{\text{Var}[X]} \]

Coefficient of variation:
\[ \frac{\sigma}{\mu} \]

Skewness:
\[ \gamma_1 = \frac{E[(X - \mu)^3]}{\sigma^3} = \frac{\mu_3}{\sigma^3} \]

Kurtosis:
\[ \gamma_1 = \frac{E[(X - \mu)^4]}{\sigma^4} = \frac{\mu_4}{\sigma^4} \]

Left truncated and shifted variable (aka excess loss variable):
\[ Y^P = X - d | X > d \]

Mean excess loss function:
\[ e_X(d) \equiv e(d) = E[Y^P] \]
\[ = E[X - d | X > d] \]
\[ = \frac{\int_{d}^{\infty} S(x)dx}{1 - F(d)} \]
\[ = \frac{E[X] - E[X \wedge d]}{1 - F(d)} \]

Higher moments of the excess loss variable:
\[ e_x^k(d) = E[(X - d)^k | X > d] \]
\[ = \frac{\int_{d}^{\infty} (x - d)^k f(x)dx}{1 - F(d)} \]
\[ = \frac{\sum_{x > d} (x - d)^k p(x)}{1 - F(d)} \]

Left censored and shifted variable:
\[ Y^L = (X - d)_+ = \begin{cases} 0 & X < d \\ X - d & X \geq d \end{cases} \]

Moments of the left censored and shifted variable:
\[ E[(X - d)^k_+] = \int_{d}^{\infty} (x - d)^k f(x)dx \]
\[ = \sum_{x > d} (x - d)^k p(x) \]
\[ = e^k(d)[1 - F(d)] \]

Limited loss:
\[ Y = (X \wedge u) = \begin{cases} X & X < u \\ u & X \geq u \end{cases} \]

Limited expected value:
\[ E[X \wedge u] = -\int_{-\infty}^{0} F(x)dx + \int_{0}^{u} S(x)dx \]
\[ = \int_{0}^{u} [1 - F(x)]dx \text{ if } X \text{ is always positive} \]

Moments of the limited loss variable:
\[ E[(X \wedge u)^k] = \int_{-\infty}^{u} x^k f(x)dx + u^k[1 - F(u)] \]
\[ = \sum_{x \leq u} x^k p(x) + u^k[1 - F(u)] \]

Moment generating functions \( m_X(t) \):
\[ m_X(t) = E[e^{tX}] \]

Sum of random variables \( S_k = X_1 + \cdots + X_k \):
\[ m_{S_k}(t) = \prod_{j=1}^{k} m_{X_j}(t) \]
Chapter 4: Classifying and Creating distributions

\( k \)-point mixture \( (\sum a_i = 1) \):

\[
F_Y(y) = a_1F_{X_1}(y) + \cdots + a_kF_{X_k}(y)
\]

\[
f_Y(y) = a_1f_{X_1}(y) + \cdots + a_kf_{X_k}(y)
\]

\[
E[Y] = a_1E[X_1] + \cdots + a_kE[X_k]
\]

\[
E[Y^2] = a_1E[X_1^2] + \cdots + a_kE[X_k^2]
\]

Tail weight:
- existence of moments \( \Rightarrow \) light tail
- hazard rate increases \( \Rightarrow \) light tail

Multiplication by a constant \( \theta, y > 0 \):

\( Y = \theta X \Rightarrow F_Y(y) = F_X\left(\frac{y}{\theta}\right) \) and \( f_Y(y) = \frac{1}{\theta}f_X\left(\frac{y}{\theta}\right) \)

Raising to a power:

\[
Y = X^\tau \Rightarrow \left\{
\begin{array}{ll}
\tau > 0 & : F_Y(y) = F_X(y^\tau) \\
& : f_Y(y) = \tau y^{\tau-1}f_X(y^\tau) \\
\tau < 0 & : F_Y(y) = 1 - F_X(y^\tau) \\
& : f_Y(y) = -\tau y^{\tau-1}f_X(y^\tau)
\end{array}
\right.
\]

\( \tau > 0 \): transformed distribution

\( \tau = -1 \): inverse distribution

\( \tau < 0 \): inverse transformed

Exponentiation:

\[
Y = e^X \Rightarrow F_Y(y) = F_X(\ln(y)) \\
f_Y(y) = \frac{1}{y}f_X(\ln(y))
\]

Mixing:

The random variable \( X \) depends upon a parameter \( \theta \), itself a random variable \( \Theta \). For a given value \( \Theta = \theta \), the individual pdf is \( f_{X|\Theta}(x|\theta) \).

\[
f_X(x) = \int f_{X|\Theta}(x|\theta)f_\Theta(\theta)d\theta
\]

- If \( X \) is a Poisson distribution with parameter \( \lambda \), and \( \lambda \) follows a Gamma distribution with parameters \( (\alpha, \theta) \), then the resulting distribution is a Negative Binomial distribution with parameters \( (r = \alpha, \beta = \theta) \)

- If \( X \) is an Exponential distribution with mean \( 1/\lambda \) and \( \lambda \) is a Gamma distribution with parameters \( (\alpha, \theta) \), then the resulting distribution is a 2-parameter Pareto distribution with parameters \( (\alpha' = \alpha, \theta' = \frac{1}{\theta}) \)

- If \( X \) is a Normal distribution \( \eta(\theta, V) \) and \( \theta \) follows a Normal distribution \( \eta(\mu, A) \), then the resulting distribution is a Normal distribution \( \eta(\mu, A + V) \)

Splicing \( (a_j > 0, \sum a_j = 1) \):

\[
f(x) = \begin{cases} 
  a_1f_1(x) & \text{if } 0 < x < c_1 \\
  \vdots \\
  a_kf_k(x) & \text{if } c_{k-1} < x < c_k 
\end{cases}
\]

Discrete probability function (pf):

\[
p_k = \text{Pr}[N = k]
\]

Probability generating function (pgf):

\[
P_N(z) = E[z^N] = p_0 + p_1z + p_2z^2 + \cdots \\
P'(1) = E[N] \\
P''(1) = E[N(N - 1)]
\]

Poisson distribution:

\[
p_k = e^{-\lambda} \frac{\lambda^k}{k!} \\
P(z) = e^{\lambda(z-1)} \\
E[N] = \lambda \\
V ar[N] = \lambda
\]

Negative Binomial distribution:

Number of failures before success \( r \) with probability of success \( p = 1/(1 + \beta) \)

\[
p_k = \binom{k+r-1}{k} \left( \frac{\beta}{1+\beta} \right)^k \left( \frac{1}{1+\beta} \right)^r \\
P(z) = [1 - \beta(z - 1)]^{-r} \\
E[N] = r\beta \\
V ar[N] = r\beta(1 + \beta)
\]

- If \( N_1 \) and \( N_2 \) are independent Negative Binomial distributions with parameters \( (r_1, \beta) \) and \( (r_2, \beta) \), then \( N_1 + N_2 \) is a Negative Binomial distribution with parameters \( (r = r_1 + r_2, \beta) \)
Exposure modifications on frequency for common distributions:

- If \( N \) is a Negative Binomial distribution with parameters \((r, \beta)\) and some events \( j \) are special with probability \( \pi_j \), then \( \tilde{N}_j \), the count of special events \( j \), is a Negative Binomial distribution with parameters \((\tilde{r}_j = r, \tilde{\beta}_j = \pi_j \beta)\)

**Geometric distribution:**
Special case of the negative binomial distribution with \( r = 1 \)

\[
p_k = \frac{\beta^k}{(1 + \beta)^{k+1}}
\]

\[
P(z) = [1 - \beta(z - 1)]^{-1} = \frac{1}{1 - \beta(z - 1)}
\]

\[
E[N] = \beta
\]

\[
Var[N] = \beta(1 + \beta)
\]

**Binomial distribution:**
Counting number of successes in \( m \) trials given a probability of sucess “\( q \)”

\[
p_k = \binom{m}{k} q^k (1 - q)^{m-k}
\]

\[
P(z) = [1 + q(z - 1)]^m
\]

\[
E[N] = mq
\]

\[
Var[N] = mq(1 - q)
\]

The \((a, b, 0)\) class:

\[
\frac{p_k}{p_{k-1}} = a + \frac{b}{k}
\]

\( a > 0 \) \( \Rightarrow \) negative binomial or geometric distribution

\( a = 0 \) \( \Rightarrow \) Poisson distribution

\( a < 0 \) \( \Rightarrow \) Binomial distribution

**Compound frequency and \((a, b, 0)\) class:**

Let \( S \) be a compound distribution of primary and secondary distributions \( N \) and \( M \) with \( p_n = Pr[N = n] \) and \( f_n = Pr[M = n] \). If \( N \) is a member of the \((a, b, 0)\) class

\[
g_k = \frac{1}{1 - a f_0} \sum_{j=1}^{k} \left( a + \frac{b}{k} \right) f_j g_{k-j}
\]

\[
g_0 = P_{prim}(f_0)
\]

\[
g_k = \frac{\lambda}{k} \sum_{j=1}^{k} j \cdot f_j g_{k-j}
\]

**Mixed Poisson models:**
If \( P(z) \) is the pgf of a Poisson distribution with rate \( \lambda \) drawn from a discrete random variable \( \Lambda \), then

\[
P(z|\lambda) = e^{\lambda(z-1)}
\]

\[
P(z) = p_\Lambda(\lambda_1)e^{\lambda_1(z-1)} + \ldots + p_\Lambda(\lambda_n)e^{\lambda_n(z-1)}
\]

**Exposure modifications on frequency:**
\( n \) policies in force. \( N_i \) claims produced by the \( j \)-th policy. \( N = N_1 + \ldots + N_n \). If the \( N_j \) are independent and identcally distributed, then the probability generating function for \( N \) is

\[
P_N(z) = [P_{N_1}(z)]^n
\]

If the company exposure grows to \( n^* \), let \( N^* \) represent the new total number of claims,

\[
P_{N^*}(z) = [P_{N_1}(z)]^{n^*} = [P_N(z)]^{n^*}
\]

**Conditional expectations:**

\[
E[X|\Lambda = \lambda] = \int x f_{X|\Lambda}(x|\lambda)dx
\]

\[
E[X] = E[E[X|\Lambda]]
\]

\[
Var[X] = E[Var[X|\Lambda]] + Var[E[X|\Lambda]]
\]

The variance of the random variable \( X \) is the sum of two parts: the mean of the conditional variance plus the variance of the conditional mean.

Exam M - Loss Models - LGD®

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**Compound frequency models:**

\( P(z) = P_{prim}(P_{sec}(z)) \)

**Exposure modifications on frequency for common distributions:**

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Parameters for ( N )</th>
<th>Parameters for ( N^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poisson</td>
<td>( \lambda )</td>
<td>( \lambda^* = \frac{n^* \lambda}{n} )</td>
</tr>
<tr>
<td>Negative Binomial</td>
<td>( r, \beta )</td>
<td>( r^* = \frac{n^<em>}{n} r, \beta^</em> = \beta )</td>
</tr>
</tbody>
</table>
Chapter 5: Frequency and Severity with Coverage Modifications

Notations:

\[ X \equiv \text{amount of loss} \]
\[ Y^L \equiv \text{amount paid per loss} \]
\[ Y^P \equiv \text{amount paid per payment} \]

Per loss variable:

\[ f_{Y^L}(0) = F_X(d) \]
\[ f_{Y^L}(y) = f_x(y + d) \]
\[ F_{Y^L}(y) = F_X(y + d) \]

Per payment variable:

\[ f_{Y^P}(y) = \frac{f_x(y + d)}{1 - F_X(d)} \]
\[ F_{Y^P}(y) = \frac{F_X(y + d) - F_X(d)}{1 - F_X(d)} \]

Relations per loss / per payment variable:

\[ Y^P = Y^L | Y^L > 0 \]
\[ E[Y^P] = \frac{E[Y^L]}{P_T|Y^L > 0} \]
\[ E[(Y^P)^k] = \frac{E[(Y^L)^k]}{P_T|Y^L > 0} \]

Simple (or ordinary) deductible \( d \):

\[ X = X \]
\[ Y^L = (X - d)_+ \]
\[ Y^P = X - d | X > d \]
\[ E[Y^L] = E[(X - d)_+] = E[X] - E[X \wedge d] \]
\[ E[Y^P] = E[X - d | X > d] = \frac{E[X] - E[X \wedge d]}{1 - F_X(d)} \]

Franchise deductible \( d \):

\[ X = X \]
\[ Y^L = 0 \text{ if } X < d, \ X \text{ if } X > d \]
\[ Y^P = X | X > d, \]
\[ Y^P_{\text{franchise}} = Y^P_{\text{ordinary}} + d \]
\[ E[Y^L] = E[X] - E[X \wedge d] + d[1 - F_X(d)] \]
\[ E[Y^P] = \frac{E[X | X > d]}{1 - F_X(d)} + d \]

Effect of deductible for various distributions:

\[ X : \text{exp}[\text{mean } \theta] \Rightarrow Y^P : \text{exp}[\text{mean } \theta] \]
\[ X : \text{uniform}[0, \omega] \Rightarrow Y^P : \text{uniform}[0, \omega - d] \]
\[ X : 2\text{Pareto}[\alpha, \theta] \Rightarrow Y^P : 2\text{Pareto}[\alpha', \theta' = \theta + \alpha] \]

Loss elimination ratio:

\[ LER_X(d) = \frac{E[X \wedge d]}{E[X]} \]

relations:

\[ c(X \wedge d) = (cX) \wedge (cd) \]
\[ c(X - d)_+ = (cX - cd)_+ \]
\[ A = (A + b) + (A - b)_+ \]

Inflation on expected cost per loss:

\[ E[Y^L] = (1 + r) \left[ E(X) - E(X \wedge \frac{d}{1 + r}) \right] \]

Inflation on expected cost per payment:

\[ E[Y^P] = \frac{(1 + r) \left[ E(X) - E(X \wedge \frac{d}{1 + r}) \right]}{1 - F_X \left( \frac{d}{1 + r} \right)} \]

\( u \)-coverage limit:

\[ Y^L = X \wedge u \]
\[ Y^P = X \wedge u | X > 0 \]
\[ E[Y^L] = E[X \wedge u] = \int_0^u [1 - F_X(x)] dx \]

\( u \)-coverage limit with inflation:

\[ E[X \wedge u] \rightarrow (1 + r)E \left[ X \wedge \frac{u}{1 + r} \right] \]

\( u \)-coverage limit and \( d \) ordinary deductible:

\[ Y^L = (X \wedge u) - (X \wedge d) \]
\[ Y^P = (X \wedge u) - (X \wedge d) | X > 0 \]
\[ = (X \wedge u) - d | X > d \]
\[ E[Y^L] = E[X \wedge u] - E[X \wedge d] \]
\[ E[Y^P] = \frac{E[X \wedge u] - E[X \wedge d]}{1 - F_X(d)} \]
$u$-coverage limit and $d$ ordinary deductible with inflation:

\[
E[Y^L] = (1 + r) \left\{ E \left[ X \wedge \frac{u}{1+r} \right] - E \left[ X \wedge \frac{d}{1+r} \right] \right\}
\]

\[
E[Y^P] = (1 + r) \left\{ \frac{E \left[ X \wedge \frac{u}{1+r} \right] - E \left[ X \wedge \frac{d}{1+r} \right]}{1 - F_X \left( \frac{d}{1+r} \right)} \right\}
\]

Coinsurance factor $\alpha$:
First calculate $\tilde{Y}^L$ and $\tilde{Y}^P$ with deductible and limits. Then apply $\alpha$

\[
Y^L = \alpha \tilde{Y}^L \\
Y^P = \alpha \tilde{Y}^P
\]

Deductible, limit, inflation and coinsurance:

\[
d^* = \frac{d}{1+r}
\]

\[
u^* = \frac{u}{1+r}
\]

\[
E[Y^L] = \alpha(1+r) \left\{ E[X \wedge u^*] - E[X \wedge d^*] \right\}
\]

\[
E[Y^P] = \alpha(1+r) \frac{E[X \wedge u^*] - E[X \wedge d^*]}{1 - F_X(d^*)}
\]

\[
E[Y^L^2] = \alpha^2(1+r)^2 \left\{ E[(X \wedge u^*)^2] - E[(X \wedge d^*)^2] - 2d^*E[X \wedge u^*] + 2d^*E[X \wedge d^*] \right\}
\]

Probability of payment for a loss with deductible $d$:

\[
v = 1 - F_X(d)
\]

Unmodified frequency:
The number of claims $N$ (frequency) does not change, only the probabilities associated with the severity are modified to include the possibility of zero payment.

Bernoulli random variable:

\[
N = \begin{cases} 
0 & @ 1 - p \\
1 & @ p
\end{cases}
\]

\[
E[N] = p \\
Var[N] = (1 - p)p
\]

Modified frequency:

$N^*$ (the modified frequency) has a compound distribution: The primary distribution is $N$, and the secondary distribution is the Bernoulli random variable $I$. The probability generating function is

\[
P_{N^*}(z) = P_N[1 + v(z - 1)]
\]

The frequency distribution is modified to include only non-zero claim amounts. Each claim amount probability is modified by dividing it by the probability of a non-zero claim. For common distributions, the frequency distribution changes as follows (the number of positive payments is nothing else than a special event with probability $\pi = v$)

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Parameters for $N$</th>
<th>Parameters for $N^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poisson</td>
<td>$\lambda$</td>
<td>$\lambda^* = v\lambda$</td>
</tr>
<tr>
<td>Binomial</td>
<td>$m, q$</td>
<td>$m^* = m, \quad q^* = vq$</td>
</tr>
<tr>
<td>Negative Binomial</td>
<td>$r, \beta$</td>
<td>$r^* = r, \quad \beta^* = v\beta$</td>
</tr>
</tbody>
</table>
Chapter 6: Aggregate Loss Models

The aggregate loss compound random variable:

\[ S = \sum_{j=1}^{N} X_j \]

\[ E[S] = E[N] \cdot E[X] \]

\[ Var[S] = E[N] \cdot Var[X] + Var[N] \cdot E[X]^2 \]

\[ S \simeq \eta(E[S], Var[S]) \text{ if } E[N] \text{ is large} \]

Probability generating function:

\[ P_S(z) = P_N \cdot P_X(z) \]

Notations:

\[ f_k = Pr[X = k] \]

\[ p_k = Pr[N = k] \]

\[ g_k = Pr[S = k] \]

Compound probabilities:

\[ g_0 = P_N(f_0) \text{ if } Pr[X = 0] = f_0 \neq 0 \]

\[ g_k = p_0 \cdot Pr[\text{sum of } X \text{'s} = k] + p_1 \cdot Pr[\text{sum of one } X = k] + p_2 \cdot Pr[\text{sum of two } X \text{'s} = k] + \ldots \]

If \( N \) is in the \((a, b, 0)\) class, then

\[ g_0 = P_N(f_0) \]

\[ g_k = \frac{1}{1 - a f_0} \sum_{j=1}^{k} \left( a + \frac{j \cdot b}{k} \right) f_j \cdot g_{k-j} \]

\( n \)-fold convolution of the pdf of \( X \):

\[ f_X^{*(n)}(k) = Pr[\text{sum of } n \text{ X's} = k] \]

\[ f_X^{*(n)}(k) = \sum_{j=0}^{k} f_X^{*(n-1)}(k-j)f_X(j) \]

\[ f_X^{*(1)}(k) = f_X(k) = f_k \]

\[ f_X^{*(0)}(k) = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise} \end{cases} \]

Recursion relations for stop-loss insurance:

\[ E[(S - d + 1)_+] = E[(S - d + 1)_+] - [1 - F_S(d)] \]

\[ E[(S - s_j)] = E[(S - s_{j+1})] + (s_{j+1} - s_j) Pr[S \geq s_{j+1}] \]

where the \( s_j \)’s are the possible values of \( S \), \( S : s_0 = 0 < s_1 < s_2 < \cdots \) and

\[ Pr[S \geq s_{j+1}] = 1 - Pr[S = s_0 \text{ or } S = s_1 \text{ or } \cdots \text{ or } S = s_j] \]

\( n \)-fold convolution of the cdf of \( X \):

\[ F_X^{*(n)}(k) = \sum_{j=0}^{k} F_X^{*(n-1)}(k-j)f_X(j) \]

\[ F_X^{*(0)}(k) = \begin{cases} 0, & k < 0 \\ 1, & k \geq 0 \end{cases} \]

Pdf of the compound distribution:

\[ f_S(k) = \sum_{n=0}^{\infty} p_n \cdot f_X^{*(n)}(k) \]

Cdf of the compound distribution:

\[ F_S(k) = \sum_{n=0}^{\infty} p_n \cdot F_X^{*(n)}(k) \]

Mean of the compound distribution:

\[ E[S] = \sum_{k=0}^{\infty} k \cdot f_S(k) = \sum_{k=0}^{\infty} [1 - F_S(k)] \]

Net stop-loss premium:

\[ E[(S - d)_+] = \int_{d}^{\infty} (x - d) f_S(x) dx \]

\[ E[(S - d)_+] = \sum_{d+1}^{\infty} (x - d) f_S(x) \]

\[ E[(S - d)_+] = E[S] - d - \sum_{x=0}^{d-1} (x - d) f_S(x) \]

\[ E[(S - d)_+] = \sum_{d}^{\infty} [1 - F_S(d)] \]

\[ E[(S - d)_+] = E[S] - \sum_{x=0}^{d-1} [1 - F_S(d)] \]

Linear interpolation \( a < d < b \):

\[ E[(S - d)_+] = \frac{b - d}{b - a} E[(S - a)_+] + \frac{d - a}{b - a} E[(S - b)_+] \]
Last Chapter: Multi-State Transition Models

Notations:

\[ Q^{(i,j)}_{n} = Pr[M_{n+1} = j|M_{n} = i] \]
\[ kQ^{(i,j)}_{n} = Pr[M_{n+k} = j|M_{n} = i] \]
\[ P^{(i)}_{n} = Q^{(i,i)}_{n} \]
\[ kP^{(i)}_{n} = Pr[M_{n+1} = \ldots = M_{n+k} = i|M_{n} = i] \]

Theorem:

\[ kQ_{n} = Q_{n} \cdot Q_{n+1} \cdot \ldots \cdot Q_{n+k-1} \]
\[ kQ_{n} = Q^{k} \text{ for a homogeneous Markov chain} \]

Inequality:

\[ kP^{(i)}_{n} = P^{(i)}_{n} \cdot P^{(i)}_{n+1} \cdot \ldots \cdot P^{(i)}_{n+k-1} \leq kQ^{(i,i)}_{n} \]

Cash flows while in states:

\[ lC^{(i)} = \text{cash flow at time } l \text{ if subject is in state } i \text{ at time } l \]
\[ kv_{n} = \text{present value of 1 paid } k \text{ years after time } t = n \]
\[ APV_{s0n}(C^{(i)}) = \text{actuarial present value at time } t = n \text{ of all future payments to be made while in state } i, \text{ given that the subject is in state } s \text{ at } t = n \]
\[ APV_{s0n}(C^{(i)}) = \sum_{k=0}^{\infty} [kQ^{(s,i)}_{n}] \cdot [n+kC^{(i)}_{n}] \cdot kv_{n} \]

Cash flows upon transitions:

\[ l+1C^{(i,j)} = \text{cash flow at time } l + 1 \text{ if subject is in state } i \text{ at time } l \text{ and state } j \text{ at time } l + 1 \]
\[ lQ^{(s,i)}_{n} \cdot Q^{(i,j)}_{n+1} = \text{probability of being in state } i \text{ at time } l \text{ and in state } j \text{ at time } l + 1 \]
\[ APV_{s0n}(C^{(i,j)}) = \text{actuarial present value at time } t = n \text{ of a cash flow to be paid upon transition from state } i \text{ to state } j \]
\[ APV_{s0n}(C^{(i,j)}) = \sum_{k=0}^{\infty} [kQ^{(s,i)}_{n} \cdot Q^{(i,j)}_{n+k}] \cdot [n+k+1C^{(i,j)}_{n+k}] \cdot k+1v_{n} \]