(P/1 Actuary Business = Valuation of Risk for Risk Management) (FM/2 Actuary Business = Valuation of Risk for Finance) (MLC Actuary Business = Modeling Risk for Finance using $N > 30$)

Book: AMfLCR (Actuarial Mathematics for Life Contingent Risk), Chapters 1-9, 10 (not 10.7), 11 (not 11.5), and 12-13.

premium: payment to insurer for insurance product
assessmentism (p.2 AMfLCR)
insurable interest: insured must suffer [financial loss] if the object insured is lost
Term insurance: lump sum on death of policyholder, subject to term (sometimes renewable)
Whole life: lump sum on death of policyholder, premium often payable only up to some max age
Endowment insurance: lump sum on death of policyholder or at end of specified term, whichever is 1st

Modern Insurance:
Universal Life
Unitized with-profit
Equity-linked

Distribution methods:
commission system
front-end load
direct marketing
pre-need insurance

Underwriting:
rating factors
proposal form
underwriting
–Preferred lives
–Normal lives
–Rated lives
–Uninsurable lives
adverse selection

SPDA
SPIA
RPDA
Joint life annuity (markov chains)
Last survivor annuity
Reversionary annuity

Income protection insurance (disability)
Critical illness insurance
Long-term care insurance

Mutual insurers
Proprietary insurers

"The primary responsibility of the life insurance actuary is to maintain the solvency and profitability of the insurer." -AMfLCR

Life insurance underwriting (3.6 AMfLCR)
Select and ultimate survival models:

aggregate survival models
select and ultimate survival model
select survival model
selected, select period, ultimate [mortality]
heterogeneity in mortality – male/female mortality differs significantly, wealthy/lower-class, individual's self-selection, the more rigorous the underwriting is the lighter the resulting experienced mortality

Mortality trends: reduction factors

The Standard Ultimate Survival Model: Makeham's with: $A = .00022$, $B = 2.7 \times 10^{-6}$, $c = 1.124$.

A_x is the expected PV of the whole life insurance benefit
We can calculate the values of A_x using backwards recursion

Variable insurance benefits using indicator random variables

net premium = risk premium = benefit premium
gross premium = office premium = expense-loaded premium
single premium: one-time payment

F.3.4 Policy Changes
- At the request of policyholder
- Admin expenses
- Anti-selection
- Liquidity risk
- Charges are made to the policyholder b/c of these risks

- Cancel policy immediately - no value is a lapse (term insurance) and positive value is a surrender (e.g. whole life) - Alter premiums - Alter benefits (e.g. change whole life to paid-up term insurance)

Non-forfeiture Options
- Reduced paid-up
- Extended term ${}_tCV_x = A^{1}_{x+t:\overline{n}|}$

Markov Chain is a multiple-state model that is memoryless Homogeneous/Non-homogeneous
Transition Matrices

Absolute rate of decrement - independent probability - pure measure of decrement - single decrement - net probability of decrement (all same - $q'^{(j)}_x$) (this is a rate, not a true probability)

Probability

$\text{Var}(X) = E[(X-\mu)^2]$

$\text{Var}(X) = E(X^2) - E(X)^2 \rightarrow$ Keep in mind $E(X^n) \neq E(X)^n$

$\text{Var}(X+a) = \text{Var}(x), \text{Var}(aX) = a^2\text{Var}(X)$

$\text{Var}(aX+bY) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X,Y)$

$\text{Var}(X) = \text{Var}(E(X|Y)) + E(\text{Var}(X|Y))$: Intuitively, given Y, X has distribution with mean $E(X|Y)$ and variance $\text{Var}(X|Y)$.

T_x future lifetime rv

$P(T_x \leq t) = P(T_0 \leq x+t | T_0 > x)$

$S_0(x+t) = S_0(x)S_x(t)$

μ_x the force of mortality of (x)

$f_x(t) = \frac{d}{dt}F_x(t) = -\frac{d}{dt}S_x(t)$

$$\mu_t = \frac{-1}{S_0(t)}\frac{d}{dt}S_0(t)$$

$$\mu_x = \frac{f_0(x)}{S_0(x)}$$

$$S_x(t) = e^{-\int_0^t \mu_{x+r}dr}$$

${}_tp_x = P(T_x > t)$

${}_{u|t}q_x = {}_up_x - {}_{u+t}p_x$

$\mathring{e}_x = E(T_x)$

$\mathring{e}_{x:\overline{n}|} = E(\min(T_x, n)) = \int_0^n t{}_tp_x\mu_{x+t}dt + n{}_np_x$

$K_x = \lfloor T_x \rfloor$

$e_x = E(K_x)$ the expected curtate future lifetime

Makeham's law: $\mu_x = A + Bc^x$ (Gompertz' eliminates constant A)

Functions for lifetime variables with means and variances:

$F_x(t) = P(T_x < t)$ $\quad S_x(t) = 1 - F_x(t)$ $\quad S_x(t) = \frac{S_0(x+t)}{S_0(x)}$

$\mu = \mathring{e}_x = E(T_x) = \int_0^\omega {}_tp_x dt$ $\quad E(T_x^2) = 2\int_0^\omega t\,{}_tp_x dt$

Under constant force of mortality: $\mathring{e}_{x:\overline{n}|} = \int_0^n e^{-\mu t}dt = \frac{1-e^{-\mu n}}{\mu}, (\mu_x = \mu)$

$l_{x+s} = l_x p_x^s$, ${}_sp_x = p_x^s$, ${}_sq_{x+t} = 1 - p_x^s$, $\mu_{x+s} = -\ln p_x$

Life tables:

$l_{x_0+t} = l_{x_0}\,{}_tp_{x_0}$

$d_x = l_x - l_{x+1}$ number of deaths over next 1yr on table for those aged x

Under UDD: $q_x = {}_sp_x\mu_{x+s}$, $l_{x+s} = l_x(1 - sq_x)$, ${}_sq_{x+t} = \frac{sq_x}{1-tq_x}$, $\mu_{x+t} = \frac{q_x}{1-sq_x}$

$A_x = E(v^{K_x+1}) = \sum_{k=0}^\infty v^{k+1}\,{}_{k|}q_x = vq_x + v^2\,{}_{1|}q_x + v^3\,{}_{2|}q_x + \ldots$

$A_x = vq_x + vp_xA_{x+1}$ $\quad K_x^{(m)} = \frac{1}{m}\lfloor mT_x \rfloor$

$$\bar{A}_x = E(e^{\delta T_x}) = \int_0^\infty e^{-\delta t}\,{}_tp_x\mu_{x+t}dt$$

$$A_x^{(m)} = \sum_{k=0}^\infty v^{\frac{k+1}{m}}\,{}_{\frac{k}{m}|\frac{1}{m}}q_x$$

$A_x^{(m)} = \frac{i}{i^{(m)}} A_x$

$$\bar{A}^1_{x:\overline{n}|} = \int_0^n e^{-\delta t}\ {}_tp_x \mu_{x+t} dt$$

$$^2\bar{A}^1_{x:\overline{n}|} = \int_0^n e^{-2\delta t}\ {}_tp_x \mu_{x+t} dt$$

Percentiles: Draw a company pain curve: For constant μ, P(SBP is sufficient)= $\left(\frac{\mu}{\mu+\delta}\right)^{\mu/\delta}$

$\bar{A}_x = \frac{i}{\delta} A_x$ $\qquad$ $\bar{A}_{x:\overline{n}|} = \frac{i}{\delta} A^1_{x:\overline{n}|} + {}_nE_x$ $\qquad$ $A_x^{(m)} = \frac{i}{i^{(m)}} A_x$

$^2\bar{A}_x = \frac{2i+i^2}{2\delta}\ {}^2A$

In general, multiply discrete case by $\frac{i}{\delta}$ to obtain continuous.

For WL and EW Ins w/ related annuities: The Most Important Identity (J. Washer): Continuous, Discrete, or m-thly:

$$A_x^{(m)} = 1 - d^{(m)} \ddot{a}_x^{(m)}$$

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k\ {}_kp_x$$

Whole life annuity-due: $\ddot{a}_x = \frac{1-A_x}{d}$ (MII)

$\ddot{a}_{x:\overline{n}|} = \frac{1-A_{x:\overline{n}|}}{d}$ (MII)

$\bar{a}_x = \frac{1-\bar{A}_x}{\delta}$ (MII)

$\ddot{a}_x^{(m)} = \frac{1-A_x^{(m)}}{d^{(m)}}$ (MII)

${}_{u|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{u}|}$

$^2A_x = 1 - {}^2d\,{}^2\ddot{a}_x = 1 - (2d - d^2)\,{}^2\ddot{a}_x$

Under constant force of mortality: $\bar{a} = (\mu + \delta)^{-1}$

$$(I\ddot{a})_x = \sum_{t=0}^{\infty} v^t (t+1)_t p_x \text{ [eqn 5.33]}$$

${}_tq_{\overline{xy}} = {}_tq_x \cdot {}_tq_y$ if T_x and T_y are independent

For $X_{\overline{xy}}$: $\mathring{e}_{\overline{xy}}$, $\bar{a}_{\overline{xy}}$, $\bar{A}_{\overline{xy}}$, or $A^1_{\overline{xy}:\overline{n}|}$:

$X_{\overline{xy}} = X_x + X_y - X_{xy}$ (4 similar eqns)

$P_{xy} = \frac{1}{\ddot{a}_{xy}} - d$

Exactly 1 status: ${}_tp^{[1]}_{\overline{xy}} = {}_tp_x + {}_tp_y - 2\,{}_tp_{xy}$

1's or 2's over one of the lives' age indicates that they die 1st or 2nd.

Const force multiple life:

$\bar{A}_{xy} = \frac{\mu_x + \mu_y}{\mu_x + \mu_y + \delta}$

$\bar{a}_{xy} = \frac{1}{\mu_x + \mu_y + \delta}$

${}_nq^1_{xy} = \frac{\mu_x}{\mu_x + \mu_y}\ {}_nq_{xy}$

Reversionary ((y) gets \$ after (x) dies): $\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}$

$$\frac{P_{x:\overline{n}|} - P_x}{P_{x:\overline{n}|}^{\ \ 1}} = {}_nV_{x:\overline{n}|} - {}_nV_x = 1 - {}_nV_x$$

Whole Life and Endowment Only:
wlew- ${}_tV_x = \frac{A_{x+t}-A_x}{1-A_x}$
wlew- $\text{Var}({}_tL) = (S+\overline{P}/\delta)^2\text{Var}(v^U)$

Accumulated cost of insurance (retrospective reserving): ${}_tk_x = \frac{A^1_{x:\overline{t}|}}{{}_tE_x}$ 1-year: ${}_1k_x = q_x/p_x$
Each P accounts for 2 items: cost of providing 1 year's DB and creation of reserve

Woolhouse, three terms: $\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu_x+\delta)$
$\mu_x \approx -\frac{1}{2}(\ln p_{x-1} + \ln p_x)$
Variance:
Annuity: $\text{Var}(\bar{Y}_x) = \frac{{}^2\bar{A}_x - \bar{A}_x^2}{\delta^2}$
Variance of an annuity?
$S \to$ Death benefit
$G \to$ Gross premium
${}_tCV \to$ Cash value

${}_tp_x^{ij} \to$ probability of changing from state i to j by time $x+t$.
${}_tp_x^{\overline{ii}} \to$ probability that state does not change from i over time t. ${}_tp_x^{\overline{ii}} \le {}_tp_x^{ii}$
${}_tq_x = {}_tp_x^{01}$ (alive-dead model)
If $\mathbf{P}^{(x)}$ is the transition matrix at age x, then ${}_tp_x^{ij}$ = index ij entry of the product $\mathbf{P}^{(x)} \cdot \mathbf{P}^{(x+1)} \cdots \mathbf{P}^{(x+t-1)}$.
μ_x^{ij} force of transition from i to j

Important formulas from FM:
$a_{\overline{n}|} = \frac{1-v^n}{i}$ $\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$ $\bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta}$ $s_{\overline{n}|} = \frac{(1+i)^n-1}{i}$ $\ddot{s}_{\overline{n}|} = \frac{(1+i)^n-1}{d}$

(incomplete) List of Formulas (FM/2–Financial Mathematics)

The following formula is given in Nicholas Mocciolo's *Making the Grade: Fourth Edition*, and formulas are given separately in ASM manual (p. 37). $1+i = \left(1+\frac{i^{(m)}}{m}\right)^m = \left(1-\frac{d^{(n)}}{n}\right)^{-n} = (1-d)^{-1}$ We note that the far left-hand and right-hand equalities are the special case of the surrounded equation where $m=n=1$.

$a_{\overline{n}|} = \frac{1-v^n}{i}$
$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$
$\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$
$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$
$s_{\overline{n}|} = \frac{(1+i)^n-1}{i}$
$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n-1}{d}$ (∼p. 114)

$\bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta}$
$\bar{a}_{\overline{n}|} = \int_0^n v^t dt$
$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1-v^n}{d^{(m)}}$

$AV = e^{\int_{t_1}^{t_2} \delta_t dt}$
$\delta_t = \frac{1}{a(t)} \frac{d}{dt} a(t)$
$a(t) = e^{\int_0^t \delta_r dr}$
If $\delta_t = \delta$ is constant: $a(t) = e^{\delta t} = (1 + i_e)^t$
and $\delta = \ln(1 + i_e)$ ($i_e \rightarrow$ effective interest rate)
$PV = \frac{a(t_1)}{a(t_2)} = e^{-\int_{t_1}^{t_2} \delta_t dt}$
(see (4), p. 199)
$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$
$(I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$
$(Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i} = \frac{s_{\overline{n+1}|} - (n+1)}{i}$
$(I\ddot{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d}$
$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}$
$(Ia)_{\overline{\infty}|} = \frac{1}{id} = \frac{1}{i} + \frac{1}{i^2}$
$(I\ddot{a})_{\overline{\infty}|} = \frac{1}{d^2}$
(see also (7), p. 217; PV= $\frac{P}{i} + \frac{Q}{i^2}$)
Perpetuity, payments of $P, P+Q, P+2Q, P+3Q, \ldots$:
PV= $\frac{P}{i} + \frac{Q}{i^2}$ (see increasing perpetuity above with $P = 1 = Q$)
Perpetuity immediate increasing then flat at n:
PV= $\frac{\ddot{a}_{\overline{n}|}}{i}$
(p. 217)
$(\bar{I}\bar{a})_{\overline{n}|} = \int_0^n tv^t = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}$
Given: $f(t)$ is the annual continuous rate of payment and δ_t is the varying force of interest:
PV= $\int_0^n f(t) e^{-\int_0^t \delta_r dr} dt$

NPV= $\sum [(A_t - L_t) v^t]$
Dollar-weighted (simple interest at $T = \{t_0, t_1, ..., u_1, u_2, ...\}$) RoR:
$A(0)(1 + t_0 i) + A(1)(1 + t_1 i) + \cdots - L_1(1 + u_1 i) - L_2(1 + u2i) - \cdots = FV$
Time-weighted: $1 + i = \left(\frac{Bal(1)}{Bal(0)}\right)\left(\frac{Bal(2)}{Bal(1.0)}\right) \ldots \left(\frac{Bal(n+1)}{Bal(n.0)}\right)$
$P_t = (Ci - Cg)v^{n-t+1}$ Interest earned on a bond: $I_t = iB_{t-1}$
Price between coupon dates: $B_{t+k} = (1+i)^k B_t = v^{1-k}(B_{t+1} + Fr)$
$1 + i' = \frac{1+i_e}{1+r}$ where r is the inflation rate
MacD= $\frac{\sum [tv^t CF_t]}{\sum v^t CF_t}$
MacD= $-(\frac{d}{d\delta} P_A)/P_A$
(Price sensitivity=[MacD]= $-\frac{P'}{P}$)
Note $\frac{\sum te^{-\delta t} A_t}{\sum e^{-\delta t} A_t} = \frac{\sum tv^t A_t}{\sum v^t A_t}$
ModD= $-(\frac{d}{di} \sum v^t A_t)/(\sum v^t A_t) = v$MacD
$(1 + i)$ModD = MacD
ModD= $-\frac{\frac{d}{di} P}{P}$
Estimate for change in price:
$\Delta P = -(\text{ModD})(P)(\Delta i)$
A little better estimate for ΔP (2nd order Taylor):

$\Delta P = -(\text{ModD})(\Delta i)(P) + \frac{1}{2}(\text{convexity})(\Delta i^2)(P)$
Convexity= $\frac{\sum t(t+1)v^{t+2}CF_t}{P_A(i)}$ ($P_A = P$, price of asset (or cash) flow at i)
Redington Immunization:
1. PV of asset flows = PV of liability flows; 2. Duration assets = Duration liabilities;
3. Conv assets > Conv of liabilities
Full Immunization:
1.,2. see Redington; 3. One asset cash inflow before and after the liability cash outflow
Immunization by exact matching (Dedication): each $A_t = L_t \forall_t$ Interest-sensitive: cash flows affected by changes in the interest rates
EffD= $\frac{P(i-h)-P(i+h)}{2hP(i)}$
EffConv= $\frac{P(i-h)+P(i+h)-2P(i)}{h^2P(i)}$

Put-Call Parity (with common strike price K, expiration at T):
$\text{Call}(K,T) - \text{Put}(K,T) = PV(F_{0,T} - K)$
Pricing a forward contract (p. 652) with spot price S_t:
No dividends:
$F_{0,T} = S_0(1+i)^T = S_0e^{rT}$
Discrete dividends:
$F_{0,T} = S_0(1+i)^T - FV(\text{dividends})$
Continuous dividends:
$F_{0,T} = S_0e^{(r-\delta)T}$

Set Theory: recall $A - B = A \cap B^C$