MFE Notes - Spring 2010 Sitting

Lesson 1 - Put-Call Parity

- Bull Spread: pays off if stock moves up in price
  - with Calls: buy C_{K_2} and sell C_{K_1}; $K_1 > K_2$
  - with Puts: buy P_{K_2} and sell P_{K_1}; $K_1 > K_2$
- Bear Spread: pays off if the stock moves down in price
  - with Calls: buy C_{K_1} and sell C_{K_2}; $K_1 > K_2$
  - with Puts: buy P_{K_1} and sell P_{K_2}; $K_1 > K_2$
- Straddle: buy a Call and a Put
  - same K $\Rightarrow$ Payoff $= |S_T - S_0|$
  - bet on volatility
  - Strangle - buy P_{K_2} and C_{K_1}; $K_1 > K_2$
- Synthetic Stock: Solve for S_0: $S_0 = e^{\delta t}\left(C - P + Ke^{-rt}\right)$
- Synthetic Treasury: Solve for Ke^{-rt}: $Ke^{-rt} = S_0e^{-\delta t} - C + P$
- Synthetic Options: Solve for C or P
- Conversion
  - Synthetically buy a T-bill
  - Lend dollars
- Reverse Conversion
  - Synthetically sell a T-bill
  - Borrow dollars

- Converting between domestic and foreign currency options:
  - A put denominated in the base currency is equivalent to some number of calls denominated in the foreign currency
  - $KP_d\left(\frac{1}{x_0}, \frac{1}{K}, T\right) = C_d(x_0, K, T)$
  - $Kx_0P_f\left(\frac{1}{x_0}, \frac{1}{K}, T\right) = C_d(x_0, K, T)$
- Bid-Ask Prices
  - The verb applie to the market-maker, not the retail customer
    - The market-maker bids the bid price when buying a share of stock
    - The market-maker asks the ask price when selling a share of stock
    - Bid Price < Ask Price

Lesson 2 - Comparing Options

- American Options:
  - Calls: $S \geq C_A \geq C_E \geq \max\left(0, F^P_{0,T}(S) - Ke^{-rT}, S_0 - K\right)$
  - Puts: $K \geq P_A \geq P_E \geq \max\left(0, Ke^{-rT} - F^P_{0,T}(S), K - S_0\right)$
- Early exercise of American Options
  - Calls:
    - lose the implicit Put
    - if non-dividend then $C_A = C_E$
    - not rational if $PV_{t,T}(Div) < K(1 - e^{-r(T-t)}) + P$
      - b/c you get stock and Divs. but pay K and lose the implicit Put
  - Puts:
    - lose the implicit Call

    * may be rationa even if no dividends
    * earn interest on K

- Different Strike Prices
  - Direction:
    * $C_1 \leq C_2$ and $P_1 \leq P_2$
    * $\frac{\partial C}{\partial K} \leq 0$ and $\frac{\partial P}{\partial K} \geq 0$
  - Slope:
    * $C_1 - C_2 \geq K_2 - K_1$ and $P_1 - P_2 \leq K_1 - K_2$
    * $\frac{\partial C}{\partial K} \geq -1$ and $\frac{\partial P}{\partial K} \leq 1$
  - Convexity:
    * $\frac{C_1 - C_2}{K_1 - K_2} \geq \frac{C_2 - C_3}{K_2 - K_3}$ and $\frac{P_1 - P_2}{K_1 - K_2} \geq \frac{P_2 - P_3}{K_2 - K_3}$
    * $\frac{\partial^2 C}{\partial K^2} \geq 0$ and $\frac{\partial^2 P}{\partial K^2} \leq 0$
      · $K_1 > K_2 > K_3$

- Strike Price Increases Over Time on a Call - suppose that a stock does not pay dividends and the strike price increases at a rate that is less than or equal to r:

$$K_T \leq K_t e^{r(T-t)}$$

The longer the call option, the more valuable it is:

$$C(S_0, K_T, T) \geq C(S_0, K_t, t) \qquad \text{for } T > t$$

If the inequality above is violated, then arbitrage is available.

That is if:

$$K_T \leq K_t e^{r(T-t)} \qquad \text{and} \qquad C(S_0, K_T, T) < C(S_0, K, T)$$

then arbitrage can be obtained with the following steps:

1. Buy the longer option and sell the shorter one
2. At time t, the shorter option is in the money, sell stock short and lend K_t at the risk-free rate

Lesson 3 - Binomial Trees - Stock, One Period

- Replicating Portfolio: B stands for bond, not borrowing; amount we lend
- $\Delta = e^{-\delta h}\left(\frac{C_u - C_d}{S(u-d)}\right)$
- $B = e^{-rh}\left(\frac{uC_d - dC_u}{u-d}\right)$
- $p^* = \frac{e^{(r-\delta)h} - d}{u-d}$
- $C = S\Delta + B$
- Multinomial Trees: Set up equations as:

$$\text{n+1 times}\begin{cases} Price_1\Delta_1 + & \cdots + & Price_n\Delta_n + & Be^{rh} = & Payoff_1 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ Price_1\Delta_1 + & \cdots & Price_n\Delta_n + & Be^{rh} = & Payoff_n \end{cases}$$

- Volatility: $d < e^{(r-\delta)h} < u$
- If tree is based on forward prices:
  - $p^* = \frac{1}{1+e^{\sigma\sqrt{h}}}$

Lesson 4 - Binomial Trees - General

- Pricing Options on Futures
  - $p^* = \frac{F_{t,t+h}/S_t - d}{u-d} = \frac{1-d}{u-d}$
  - $\Delta = \frac{C_u - C_d}{F(u-d)}$
  - $B = C = e^{-rh}\left(\frac{uC_d - dC_u}{u-d} + \frac{C_u - C_d}{u-d}\right)$
    - $B = C$ because there is no initial cost

Lesson 5 - Risk-Neutral Pricing

- $Se^{\alpha h} = puSe^{\delta h} + (1-p)dSe^{\delta h}$
  - $\Rightarrow p = \frac{e^{(\alpha-\delta)h}-d}{u-d}$
- $e^{\gamma h} = \frac{S\Delta}{S\Delta+B}e^{\alpha h} + \frac{B}{S\Delta+B}e^{rh}$
  - $\Rightarrow$ taking $E(.) \Rightarrow C = e^{-\gamma h}(pC_u + (1-p)C_d)$
- $Ce^{\gamma h} = S\Delta e^{\alpha h} + Be^{rh}$
  - $\gamma \sim$ discount rate for an option
  - $\gamma_{Call} > \alpha > r > \gamma_{Put}$
- These results are equivalent to using p^*
  - $\Rightarrow C = e^{-\gamma h}(pC_u + (1-p)C_d) = e^{-rh}(p^*C_u + (1-p^*)C_d)$
    * If $C_d = 0 \Rightarrow e^{-\gamma h}p = e^{-rh}p^*$
- Risk-Neutral Pricing and Utility(annual not cont. rates)
  - $U_i \sim$ current value of \$1 paid at the end of one year when the price of the stock is in state i
    * $U_H \leq U_L$ because of decling MU
    * if risk-neutral: $U_H = U_L = \frac{1}{1+r}$
  - $C_i \sim$ cash flow of the stock at the end of one year in state i
  - $Q_i \sim$ the current value of \$1 paid at the end of one year only if the price of the stock is C_i
- Important Formulas:
  - $Q_H = pU_H \qquad\qquad Q_L = (1-p)U_L$
  - $Q_H + Q_L = \frac{1}{1+r}$
  - $C_0 = pU_HC_H + (1-p)C_LQ_L = Q_HC_H + Q_LC_L$
  - $1+\alpha = \frac{pC_H+(1-p)C_L}{C_0} = \frac{pC_H+(1-p)C_L}{pU_HC_H+(1-p)U_LC_L} = \frac{pC_H+(1-p)C_L}{Q_HC_H+Q_LC_L}$
  - $p^* = \frac{pU_H}{pU_H+(1-p)C_L} = \frac{Q_H}{Q_H+Q_L}$

- $\Rightarrow$ solve for $p \Rightarrow p = \frac{p^* U_L}{p^* U_L + (1-p^*)U_H}$

Lesson 6 - Binomial Trees: Misc. Topics

- Understanding early exercise of Options
  - Compare $S(1-e^{-\delta t})$ vs. $K(1-e^{-rt})$
    - depends on Call vs. Put
- Lognormality and Alternative Trees
  - if annual volatility is $\sigma \Rightarrow$ monthly is $\frac{\sigma}{\sqrt{12}}$
  - Cox-Ross-Rubinstein Tree
    - centered on 1
    - $u = e^{\sigma\sqrt{h}}$ $\quad d = e^{-\sigma\sqrt{h}}$
  - Lognormal / Jarrow-Rudd Tree
    - centered on $e^{r-\delta-\frac{1}{2}\sigma^2}$
- Estimating Volatility
  - $\hat{\sigma} = \sqrt{p}\sqrt{\frac{n}{n-1}\left(\frac{\sum x_i^2}{n} - \bar{x}^2\right)}$
  - use $ln\left(\frac{S_t}{S_{t-1}}\right)$ for data points

Lesson 7 - Modeling Stock Price with the Lognormal Distribution

- If $X \sim N(\mu, \sigma^2)$ then $Y = e^X \sim LogNormal(\mu, \sigma)$
- Properties
  - $E(Y) = e^{m+\frac{1}{2}v^2}$
  - $V(Y) = e^{2m+v2}\left(e^{v^2} - 1\right)$

- mode $= e^{m-v^2}$
  - * $m = \mu t = \left(\alpha - \delta - \frac{1}{2}\sigma^2\right) t$
  - * $v = \sigma\sqrt{t}$

- Lognormal Confidence Intervals: Assume that the stock prices are lognormally distributed:
  $ln\left(\frac{S_T}{S_t}\right) \sim N\left[(\alpha - \delta - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t)\right] \quad T > t$
  The $(1-p)$ confidence interval is:
  $$Pr\left(S_T^L < S_T < S_T^U\right) = 1 - p$$
  The lower and upper stock prices defining the confidence interval are:
  - $S_T^L = S_t e^{(\alpha-\delta-\frac{1}{2}\sigma^2)(T-t)+|\sigma_Z^L|\sqrt{T-t}}$
  - $S_T^U = S_t e^{(\alpha-\delta-\frac{1}{2}\sigma^2)(T-t)+|\sigma_Z^U|\sqrt{T-t}}$
    - * where $Pr(z < z^L) = \frac{p}{2}$ and $Pr(z > z^U) = \frac{p}{2}$
    - * σ can be given as negative, that's why abs. value signs are there

- Jensen's Inequality: $E(g(X)) \geq g(E(X))$
  - $E(X^2) \geq (E(X))^2$

- Pricing European Options using the Lognormal Model
  - $Pr(S_T < K) = N(-\hat{d}_2) \qquad Pr(S_T > K) = N(\hat{d}_2)$
  - $E(X|Y) = \frac{PE(X|Y)}{Pr(Y)}$
  - $PE[S_T|S_T > K] = S_0 e^{(\alpha-\delta)t} N(\hat{d}_1)$
  - $PE[S_T|S_T < K] = S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)$
  - $PE[K|S_T < K] = KN(-\hat{d}_2)$
  - $PE[K|S_T > K] = KN(\hat{d}_2)$
  - $E[K - S_T|S_T < K] = \frac{PE[K-S_T|S_T<K]}{Pr(S_T<K)} = \frac{KN(-\hat{d}_2) - S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)}{N(-\hat{d}_2)}$
  - $E[S_T - K|S_T > K] = \frac{PE[S_T-K|S_T>K]}{Pr(S_T>K)} = \frac{S_0 e^{(\alpha-\delta)t} N(\hat{d}_1) - KN(\hat{d}_2)}{N(\hat{d}_2)}$

- Expected Payoff
  - Call: $E[max(0, S_T - K)] = S_0 e^{(\alpha-\delta)t} N(\hat{d}_1) - KN(\hat{d}_2)$
  - Put: $E[max(0, K - S_T)] = KN(-\hat{d}_2) - S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)$
- Expected Value
  - $E[S_T | S_0] = S_0 e^{\left(\mu + \frac{1}{2}\sigma^2\right)t}$

Lesson 8 - Fitting Stock Prices to a Lognormal Distribution

- Estimate using $ln\left(\frac{S_t}{S_{t-1}}\right)$ as data points
- Annual Return: $\hat{\alpha} = \hat{\mu} + \frac{1}{2}\hat{\sigma}^2$
- Drawing a Normal Probability Plot in 5 Easy Steps
  1. Sort the data into order statistics, from smallest to largest
  2. Convert the order statistics into quantiles by matching them with the appropriate cumulative probabilities
  3. Match each cumulative probability with its corresponding z-value
  4. Graph the points with the quantiles on the horizontal axis and the z-values on the vertical axis
  5. Draw a straight line through the 25% and 75% quantiles

Lesson 9 - The Black-Scholes Formula

- Black-Scholes Formula for Options on Futures
  - $C = Fe^{-rt}N(d_1) - Ke^{-rt}N(d_2)$
  - $P = Ke^{-rt}N(-d_2) - Fe^{-rt}N(-d_1)$
  - $d_1 = \frac{ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2}{\sigma\sqrt{t}}$
  - $d_2 = d_1 - \sigma\sqrt{t}$

- The futures period affects the forward price of the stock but does not affect the option price in any other way

Lesson 10 - The Black-Scholes Formula: Greeks

- $\Delta_C - \Delta_P = e^{-\delta t}$
  - $\Delta_C = e^{-\delta t} N(d_1)$
  - S-shaped
- $\Gamma_C = \Gamma_P$
  - Symmetric hump, peak to the left of K (further with higher t)
- $Vega_C = Vega_P$
  - Asymmetric hump; peak similar to Γ
- $C_t - P_t = -\delta S e^{-\delta t} + rKe^{-rt}$
  - $\Theta = -\frac{1}{365} C_t \Rightarrow \Theta_C - \Theta_P = \frac{\delta S e^{-\delta t} - rKe^{-rt}}{365}$
  - Upside-down hump; almost always < 0 unless far in the money
- $\rho_C - \rho_P = .01tKe^{-rt}$
  - Assuming ρ expressed in terms of % points
  - Increasing curve; positive for C, negative for P
- $\Psi_C - \Psi_P = -.01tSe^{-\delta t}$
  - Assuming Ψ expressed in terms of % point
  - Decreasing curve; negative for C, positive for P
- Elasticity and Related Concepts
  - $\Omega = \frac{\epsilon \Delta / C}{\epsilon / S} = \frac{S\Delta}{C}$
  - $\sigma_{option} = \sigma_{stock} |\Omega|$

  - $\gamma - r = \Omega(\alpha - r) \Rightarrow \frac{\gamma - r}{\sigma_{option}} = \frac{\Omega(\alpha - r)}{\Omega\sigma_{stock}} = \frac{\alpha - r}{\sigma_{stock}}$

- Greek for Portfolio: Σ of the greeks
- Elasticity for Portfolio: Wtd. Average of the Ω's

Lesson 11 - The Black-Scholes Formula: Applications and Volatility

- Purchase a t-day call and hold it for 1 day. Profit =
  1. Change in call premium $(C_{t-1} - C_t)$
  2. Lost interest $(e^{r/365}C_t - C_t)$

  - $C_{t-1} - C_t e^{r/365} \sim$ Difference of 1. and 2.

- Volatility
  - Black-Scholes assumes σ is constant
  - Implied Volatility: volatility that reproduces the price of an option in a pricing model.
    * Common patterns for implied equity volatilities:
      1. Decreases with strike price
      2. Flatter curve for longer time until expiration
  - Volatility Skew: refers to the fact that the implied volatility is not constant across strike prices
    * implied volatility declined as time to expiry increased
    * implied volatility decreased as K increased
    * in-the-money call has higher volatility than an out-of-the-money call
    * in-the-money put has lower volatility than an out-of-the-money put
- $\Delta = \frac{\text{change in } C}{\text{change in } S}$

Lesson 12 - Delta Hedging

- Overnight Profit on a Delta-Hedged Portfolio
  - Profit $= -(C_1 - C_0) + \Delta(S_1 - S_0) - (e^{\frac{r}{365}} - 1)(\Delta S_0 - C_0)$
- Break even for Market Maker
  - $S \pm S\sigma\sqrt{h}$
- Delta-gamma-theta approximation
  - $C_1 = C_0 + \Delta\epsilon + \frac{1}{2}\Gamma\epsilon^2 + \theta h$
    * $\epsilon = S_h - S_0$
- Black-Scholes Equation
  - $rC = S\Delta(r - \delta) + \frac{1}{2}\Gamma S^2\sigma^2 + \theta h$
- Greeks for Binomial Trees
  - $\Delta(S,0) = e^{-\delta h}\left(\frac{C_u - C_d}{S(u-d)}\right)$
  - $\Gamma(S,h) \approx \Gamma(S,0) = \frac{\Delta(Su,h) - \Delta(Sd,h)}{S(u-d)}$
  - $C(Sud,2h) = C(S,0) + \Delta(S,0)\epsilon + \frac{1}{2}\Gamma(S,0)\epsilon^2 + 2h\theta(S,0)$
    * $\Rightarrow \theta(S,0) = \frac{C(Sud,2h) - C(S,0) - \Delta(S,0)\epsilon - \frac{1}{2}\Gamma(S,0)\epsilon^2}{2h}$
      · $\epsilon = Sud - S$
- Reheding
  - Variance of the return for a single period
    * $Var[R_{h,i}] = \frac{1}{2}(S^2\sigma^2\Gamma h)^2$
  - If we re-hedge every h (measured per year)
    * Annual Variance of Return $= \frac{1}{h}Var[R_{h,i}] = \frac{1}{2}(S^2\sigma^2\Gamma)^2 h$
- Misc. Notes
  - Sell Call $\Rightarrow$ Buy Stock

- Sell Put ⇒ Sell Stock

Lesson 13 - Asian, Barrier and Compound Options

- Maxima and Minima
  - $max(S, K) = S + max(0, K - S) = K + max(0, S - K)$
  - $max(cS, cK) = c \cdot max(S, K)$
  - $max(S, K) + min(S, K) = S + K$
- Compound Options
  - $CoC - PoC = C - x_0 e^{-rt_1}$
  - $CoP - PoP = P - x_0 e^{-rt_1}$
- American options on Stock with 1 discrete dividend
  - $C_A = S_0 - Ke^{-rt_1} + CoP(S, K, D - K(1 - e^{-r(T-t_1)}), t_1, T)$
- Asian Options - ignore initial price

Lesson 14 - Gap, Exchange and Other Options

- All-or-nothing Options
  - $S|S > K = S_0 e^{(r-\delta)T} N(d_1)$
  - $S|S < K = S_0 e^{(r-\delta)T} N(-d_1)$
  - $c|S > K = ce^{-rT} N(d_2)$
  - $c|S < K = ce^{-rT} N(-d_2)$
  - Delta for all-or-nothing options
    * $\frac{\partial N(d_i)}{\partial S} = \frac{e^{-\frac{d_i^2}{2}}}{S\sigma\sqrt{2\pi T}}$
- Gap Options

- Remember that $S_T >$ trigger for Calls and $S_T <$ trigger for Puts
- Put-Call Parity applies
- If two otherwise identical gap options have different strike prices, then use linear interpolation to find the price of a third otherwise identical gap option with a different strike price.

- Exchange Options
  - volatility measures the variance of rate of return (not the dollar return)i.e. 2 shares have the volatility as 1 share

- Chooser Options
  - Derivation

$$
\begin{aligned}
V_t &= max(C(S,K,T-t), P(S,K,T-t)) \quad (1)\\
&= C(S,K,T-t) + max(0, P(S,K,T-t) - C(S,K,T-t) \quad (2)\\
&= C(S,K,T-t) + max(0, Ke^{-r(T-t)} - Se^{-\delta(T-t)}) \quad (3)\\
&= C(S,K,T-t) + e^{-\delta(T-t)} \cdot max(0, Ke^{-(r-\delta)(T-t)} - S) \quad (4)
\end{aligned}
$$

@ $t_0 \Rightarrow V_0 = C(S,K,T) + e^{-\delta(T-t)} \cdot P(S, Ke^{-(r-\delta)(T-t)}, t)$

- Forward Start Options
  - Purchase a call @ t with $K = cS_t$ expiring @ T, then the value of the forward start option is:
    - $V = Se^{-\delta T}N(d_1) - cSe^{-r(T-t)-\delta t}N(d_2)$
      - d_i are computed using $T-t$ as time to expiry

Lesson 15 - Monte Carlo Valuation

- Generating LogNormal random numbers
  1. Let $z_j = \sum_{i=1}^{12} u_i - 6$ where $u_i \in U[0,1]$
  2. Let $z_j = N^{-1}(u_j)$

- Use r to discount when pricing options
- Use α for true expected payoffs
- Control Variate Method
  - Let $X^* = \bar{X} + (E(Y) - \bar{Y})$, $Y \sim$ control variate
    * $\Rightarrow V(X^*) = V(\bar{X}) + V(\bar{Y}) - 2Cov(\bar{X}, \bar{Y})$
    * Always use sample variance / covariance formula
  - Boyle modification:
    * $X^* = \bar{X} + \beta(E(Y) - \bar{Y})$
      · $\Rightarrow V(X^*) = V(\bar{X}) + \beta^2 V(\bar{Y}) - 2\beta Cov(\bar{X}, \bar{Y})$
      · Optimal value for $\beta = \frac{Cov(\bar{X}, \bar{Y})}{V(\bar{Y})}$
      · Variance becomes: $V(X^*) = V(\bar{X})\left(1 - \rho^2_{\bar{X},\bar{Y}}\right)$
- Other Variance Reduction Techniques
  - Antithetic Variates: for every u_i, use $1 - u_i$
  - Stratified Sampling: break sampling space into strata and then scale uniform #s to be in these strata
    * If you had 4 strata: $[0, .25), \ldots, [.75, 1)$ then generate sets of 4 u_i on $[0, 1)$, multiply all 4 by .25, put the first in $[0, .25)$, add .25 to 2nd number, etc.
  - Latin Hypercube Sampling
  - Importance Sampling
  - Low Discrepancy Sequences

Lesson 16 - Brownian Motion

- Random Walk

1. $X(0) = 0$

2. For $t > 0$, if $X(t-1) = k$, then $X(t) = \begin{cases} k+1, & \text{with } p = \frac{1}{2} \\ k-1, & \text{with } p = \frac{1}{2} \end{cases}$

3. Memoryless.

   – $Pr(X(t+u) = l | X(t) = k) = Pr(X(u) = l - k)$

4. $X(t)$ is random, distance traversed is not.

   – Sum of the squares of the movement is t

5. $X(t) \sim Bin\left(t, \frac{1}{2}\right)$

- <u>Brownian Motion</u>

  – Move $\sqrt{h}$ per h units of time and take $\lim_{h \to 0}$

    * $\Rightarrow$ Cont. Random Walk and Binomial $\to$ Normal

  – Properties

    1. $Z(0) = 0$
    2. $Z(t+s)|Z(t) \sim N(Z(t), s)$
    3. $Z(t+s_1) - Z(t)$ is independent of $Z(t) - Z(t - s_2)$
    4. $Z(t)$ is cont. in t

  – <u>Expected Values</u> Under Pure Brownian Motion

    * $E[Z(t)] = 0$
    * $E[Z(t+h)|Z(t)] = Z(t)$
    * $E[Z(t+h) - Z(t)] = 0$
    * $E[dZ(t)] = 0$
    * $E[dZ(t)|Z(t)] = 0$
    * $E\left[(Z(t))^2\right] = t$

  * $E\left[(dZ(t))^2\right] = dt$
  * $E[Z(t)Z(s)] = Min(t, s)$

- Variances under Pure Brownian Motion
  * $V[Z(t)] = t$
  * $V[Z(t+h)|Z(t)] = h$
  * $V[Z(t+h) - Z(t)] = h$
  * $V[dZ(t)] = dt$
  * $V[dZ(t)|Z(t)] = dt$

- is a diffusion process - cont. process in which the absolute value of the R.V. tends to get larger
- is a martingale - process $X(t)$ for which $E[X(t+s)|X(t)] = X(t)$
  * ABM and GBM are martingales iff they have zero drift

• Arithmetic Brownian Motion
  - $X(t) = \alpha t + \sigma Z(t)$
  - $X(t+s) - X(t) \sim N(\mu s, \sigma^2 s)$
  - $X(t+s)|X(t) \sim N(X(t) + \mu s, \sigma^2 s)$

• Geometric Brownian Motion
  - If $ln\left(\frac{X(t)}{X(0)}\right) \sim N(\mu t, \sigma^2 t)$ then $X(t) - X(0) \sim$ LogNormal
    * Mean $= e^{(\mu + \frac{1}{2}\sigma^2)t}$
    * Variance $= e^{(2\mu+\sigma^2)t}(e^{\sigma^2 t} - 1)$

• To go from GBM to ABM, you must subtract $\frac{1}{2}\sigma^2$
  - When dealing with probabilities, you must convert to ABM

- $Var(ln(S(t))|S(0)) = Var(ln(F_{0,T}(S))) = Var(ln(F^P_{0,T}(S)))$
- Forms of BM
  - GBM: $\frac{dS}{S} = (\alpha - \delta)dt + \sigma dZ$
  - ABM: $d(ln(S)) = (\alpha - \delta - \frac{1}{2}\sigma^2)dt + \sigma dZ$
- When you add δ to total return (for Sharpe Ratio), only add to S, not C
- Portfolio Returns: Suppose that a portfolio P consists of 2 assets, A and B. If x is the percentage of the portfolio is invested in A and $(1-x)$ is the percentage invested in B, then the instantaneous change in the price of the portfolio is:
  $$\frac{dP(t)}{P(t)} = x\frac{dA(t)}{A(t)} + (1-x)\frac{dB(t)}{B(t)}$$
  To find the instantaneous return on the portfolio, include the dividends.
  - Instantaneous Return on Portfolio
    * $\frac{dP(t)}{P(t)} + (x\delta_A + (1-x)\delta_B)dt = x\left[\frac{dA(t)}{A(t)} + \delta_A\right] + (1-x)\left[\frac{dB(t)}{B(t)} + \delta_B\right]$

Lesson 17 - Itô's Lemma

- $dC = C_S dS + \frac{1}{2}C_{SS}(dS)^2 + C_t dt$
- Multiplication rules: All $\rightarrow 0$ except $(dZ)^2 = dt$
- The Black-Scholes Equation
  - $rC = S\Delta(r - \delta) + \frac{1}{2}\Gamma S^2\sigma^2 + \theta h$
- Sharpe Ratio(Only works for GBM)
  - $\phi = \frac{\alpha - r}{\sigma}$
    * α is total return (includes δ)

- For 2 Itô processes with the same dZ, the Sharpe Ratios are equal

- Problems which give 2 processes, Prices and ask how much should be allocated to each process. Such as:

  1. $\frac{dS_1}{S_1} = \alpha_1 dt + \sigma_1 dZ$ and $\frac{dS_2}{S_2} = \alpha_2 dt + \sigma_2 dZ$
  2. x shares of S_1 and y shares of S_2, $r = r$

     (a) Solve $S_1 \cdot x \cdot \alpha_1 + S_2 \cdot y \cdot \alpha_2 = (S_1 \cdot x + S_2 \cdot y)r$

     (b) If you know x and need y, look @ $\frac{\sigma_1}{\sigma_2}$, that's ratio of value of S_2 you need to buy/sell. Since S_1 costs $S_1 \cdot x$ then you need to buy/sell $S_1 \cdot x \left(\frac{\sigma_1}{\sigma_2}\right) = y$

- <u>CAPM</u>: $\frac{\alpha_i - r}{\sigma_i} = \rho_{i,M}\left(\frac{\alpha_M - r}{\sigma_M}\right)$

  - $\phi_i = \rho_{i,M}\phi_M$

- <u>Risk-Neutral Processes</u>

  - True Itô Process: $dS = (\alpha - \delta)dt + \sigma dZ$
  - Risk-Neutral Itô Process: $dS = (r - \delta)dt + \sigma d\tilde{Z}$

    * $d\tilde{Z} = dZ + \eta dt$
      · where $\eta = \frac{\alpha - r}{\sigma}$
    * $E^*[\tilde{Z}(T)] = 0$
    * $E^*[Z(T)] = \left(\frac{r-\alpha}{\sigma}\right)T$
    * $E[Z(T)] = 0$
    * $E[\tilde{Z}(T)] = \left(\frac{\alpha - r}{\sigma}\right)T$

- <u>Valuing a Forward on S^a</u>

  - $E[S(T)^a] = S_0^a e^{[a(\alpha-\delta)+\frac{1}{2}\sigma^2 a(a-1)]T}$

- $F_{0,T}(S^a) = S_0^a e^{[a(r-\delta)+\frac{1}{2}\sigma^2 a(a-1)]T}$
- $F_{0,T}^P(S^a) = e^{-rT} \cdot F_{0,T}(S^a)$

- Itô Process for S^a
  - If $C = S^a$ and $\frac{dS}{S} = (\alpha - \delta)dt + \sigma dZ$ then
    * $\frac{dC}{C} = \underbrace{\left(a(\alpha-\delta) + \frac{1}{2}\sigma^2 a(a-1) + \delta^*\right)}_{\gamma} dt + \sigma a dZ$
      · $\delta^* \sim$ derivative's dividend yield
    * $\Rightarrow$ Sharpe Ratios $= \frac{\gamma - r}{a\sigma} = \frac{\alpha - r}{\sigma}$
      · $\Rightarrow \gamma = a(\alpha - r) + r$

- Stochastic Integration
  - Regular Calculus Rules Apply (e.g. FTC)
    1. $\int_0^T dZ(t) = Z(T) - Z(0) \sim N(0,T)$
    2. $\int_0^T (dZ(t))^2 = \int_0^T dt = T - 0 = T$
    3. $\int_0^T (dZ(t))^n = 0, n > 2$
    4. $S(t) = \int_0^T sZ(s)ds \Rightarrow dS = t \cdot Z(t)dt$
    5. $S(t) = \int_0^T tdZ(s) \Rightarrow dS = dt\left(\int_0^T dZ(s)\right) + t\left(\int_0^T dZ(s)\right)' = Z(t)dt + tdZ(s)$

- Ornstein-Uhlenbeck Process
  - DE: $dX = \lambda(\alpha - X(t))dt + \sigma dZ$
  - Integral: $X(t) = X_0 e^{-\lambda t} + \alpha(1 - e^{-\lambda t}) + \sigma \int_0^t e^{\lambda(s-t)} dZ(s)$

- Misc. Notes

  - volatility of S^n is $n \cdot \sigma$ (remember when working with Black-Scholes)

Lesson 18 - Binomial Tree Models for Interest Rates

- $F_{t,T}(P(T, T+s)) \sim$ forward price @ t for an agreement to buy a bond @ T maturing @ $T+s$
  - $F_{t,T}(P(T, T+s)) = \frac{P(t,T+s)}{P(t,T)}$
- Binomial Trees
  - don't necessarily recombine
  - risk-neutral probs. are given
  - list out all paths and discount by that factor
- The Black-Derman-Toy model
  - Bond Price $= \frac{1}{(1+R)^n}$
  - Ratio between interest rates @ successive nodes is constant
    * it is $e^{2\sigma_t\sqrt{h}}$
  - $\sigma = \frac{\frac{1}{2}ln\left(\frac{R_u}{R_d}\right)}{\sqrt{h}}$
  - $\frac{\text{2 year Bond Price}}{\text{1 year Bond Price}} = \frac{1}{2}\left(\frac{1}{1+R_1} + \frac{1}{1+R_1 e^{2\sigma}}\right)$
- Pricing Forwards using BDT
  - use annual not cont. compouding
- Pricing Caps using BDT

- Discount difference due to cap by the discount rate appropriate to the beginning of the year
- Cap pays $max\left(0, \frac{R_T - K_R}{1+R_T}\right)$
- For multiple year trees, start @ end and calculate the value then weigh the results and add in the additional cap values as you move to t_0

Lesson 19 - The Black Formula for Bond Options

- $C(F, P(0,T), \sigma, T) = P(0,T)(FN(d_1) - KN(d_2))$
- $P(F, P(0,T), \sigma, T) = P(0,T)(KN(-d_2) - FN(-d_1))$
  - where $d_1 = \frac{ln\left(\frac{F}{K}\right)+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$
- Pricing Caps with the Black Formula
  - $(1 + K_R)$ Puts with strike price $\frac{1}{1+K_R}$
  - Calculate @ each node and then add together and multiply the sum by $(1 + K_R)$

Lesson 20 - Eq. Interest Rate Models: Vasicek and Cox-Ingersoll-Ross

- Eq. Models - Theory
  - $dr = a(r)dt + \sigma(r)dZ$ and $\frac{dP}{P} = \alpha(r,t,T)dt - q(r,t,T)dZ$
  - by Itô's, $dP = P_r dr + \frac{1}{2}P_{rr}(dr)^2 + P_t dt$
    - $\Rightarrow dP = \alpha(r,t,T)dt - q(r,t,T)dZ$, where
      - $\alpha(r,t,T) = \frac{1}{P}\left(a(r)P_r + \frac{1}{2}\sigma^2(r)P_{rr} + P_t\right)$
      - $q(r,t,T) = \frac{1}{P}P_r\sigma(r)$

- Black-Scholes equation for Bonds
  - $rP = (a(r) + \sigma(r)\phi)P_r + \frac{1}{2}\sigma^2(r)P_{rr} + P_t$
- Risk Premium $= \sigma(r)\phi$
- To go to Risk Neutral, add $\sigma(r)\phi$
  - $dr = (a(r) + \sigma(r)\phi)dt + \sigma(r)d\tilde{Z}$
  - $\tilde{Z}(t) = Z(t) - \phi$
- The Rendelman-Barter Model(GBM)
  - $dr = ardt + \sigma rdZ$
  - Interest rates cannot go negative (+)
  - Volatility is proportional to interest rate (+)
  - Interest rates can get arbitrarily high, no mean reversion (-)
  - Determine probabilities like with any GBM problem
- The Vasicek Model
  - $dr = a(b - r)dt + \sigma dZ$
  - There is mean reversion (+)
  - Volatility is constant (-)
  - Interest rates can go negative (-)
  - DE: $(a(b - r) + \sigma\phi)P_r + \frac{1}{2}\sigma^2 P_{rr} + P_t = rP$
  - $P(r, t, T) = A(t, T)e^{-B(t,T)r}$
  - $a \neq 0$

    * $A(t,T) = e^{\bar{r}[B-(T-t)]-B^2\frac{\sigma^2}{4a}}$
    * $B(t,T) = \frac{1-e^{-a(T-t)}}{a}$
    * $\bar{r} = b + \sigma\frac{\phi}{a} - \frac{1}{2}\left(\frac{\sigma}{a}\right)^2$

  - $a = 0$

    * $A(t,T) = e^{\frac{1}{2}\sigma\phi(T-t)^2+\sigma^2\frac{(T-t)^3}{6}}$
    * $B(t,T) = T - t$

  - $\Delta = P_r = -BP$

  - $\Gamma = P_{rr} = B^2P$

- The Cox-Ingersoll-Ross Model

  - $dr = a(b-r)dt + \sigma\sqrt{r}dZ$

  - Interest rates cannot go negative (+)

  - Volatility varies with interest rate (+)

  - There is mean reversion (+)

  - $\phi\sigma = \bar{\phi}r$

  - DE: $[a(b-r) + \bar{\phi}r]P_r + \frac{1}{2}\sigma^2P_{rr} + P_t = rP$

  - $P(r,t,T) = A(t,T)e^{-B(t,T)r}$

  - $A(t,T) = \left[\frac{2\gamma e^{(a-\bar{\phi}+\gamma)(T-t)/2}}{(a-\bar{\phi}+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{\frac{2ab}{\bar{\sigma}^2}}$

  - $B(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(a-\bar{\phi}+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}$

    * where $\gamma = \sqrt{(a-\bar{\phi})^2 + 2\bar{\sigma}^2}$

- Misc. Notes

  - $q(r,t,T) = -\frac{P_r}{P}\sigma(r) = B\sigma(r)$

  - Vasicek
    - $\alpha(r,t,T) = -a(b-r)B + \frac{1}{2}\sigma^2 B^2 + \frac{P_t}{P}$

- Delta Hedging
  - Duration Hedge: $N = -\frac{T_1 P(r,0,T_1)}{T_2 P(r,0,T_2)}$
  - Delta Hedge: $N = -\frac{P_r(r,0,T_1)}{P_r(r,0,T_2)}$
    - where numerator is what you are hedging

- Delta-Gamma-Theta Approximation
  - $P(r+\epsilon, 0, t+h) = P(r,0,t) + \Delta\epsilon + \frac{1}{2}\Gamma\epsilon^2 + \theta h$